The Long-Term Swap Rate and a General Analysis of Long-Term Interest Rates

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Abstract

We introduce here for the first time the long-term swap rate, characterised as the fair rate of an overnight indexed swap with infinitely many exchanges. Furthermore we analyse the relationship between the long-term swap rate, the long-term yield, see [3], [4], and [25], and the long-term simple rate, considered in [7] as long-term discounting rate. We finally investigate the existence of these long-term rates in two term structure methodologies, the Flesaker-Hughston model and the linear-rational model.

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1 Introduction

The modelling of long-term interest rates is a very important topic for financial institutions investing in securities with maturities that have a long time-horizon such as life insurances or infrastructure projects. Most articles focusing on long-term interest rate modelling examine the long-term yield, defined as the continuously compounded spot rate where the maturity goes to infinity, as discounting rate for these products, cf. [3], [4], [12], [25], or [33]. An important result which characterises the long-term yield is the Dybvig-Ingersoll-Ross (DIR) theorem, which states that the long-term

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yield is a non-decreasing process. It was first shown in [12] and then discussed in [20], [23], [24], [26], and [31]. According to [7] the DIR theorem ultimately implies that discounted cashflows with higher time-to-maturity are over-penalised, so that the use of this long-term interest rate becomes unsuitable for the valuation of projects having maturity in a distant future. To overcome this issue, in [7] the authors propose to use for discounting the long-term simple rate, which is defined as the simple spot rate with an infinite maturity. Motivated by this ongoing discussion in the literature, we investigate in this paper alternative long-term interest rates.

We introduce here for the first time the long-term swap rate, which we define as the fair fixed rate of a fixed to floating swap with infinitely many exchanges. To the best of our knowledge, there has not been any attempt in the literature to study the long-term swap rate so far. Our interest in the long term swap rate is motivated by the observation that some financial products may involve the interchange of cashflows on a possibly unlimited time horizon. This is the case of some kind of contingent convertible (CoCo) bonds, which became popular after the financial crisis in 2008. Such products are debt instruments issued by credit institutes which embed the option for the bank to convert debt into equity, typically in order to overcome the situation where the bank is not capitalised enough (cf. [1], [6], [11], [16], and [17]). In the course of the crisis the importance of CoCo bonds for financial institutions to maintain a certain level of capital was pointed out in [2]. In [10], the increase in the use of them in systemically relevant financial institutions was one of three main points that should be realised in the aftermath of the crisis to strengthen the financial system.

As reported in [6] the value of these instruments may be decomposed as a portfolio consisting of plain bonds and exotic options. A valuation method for CoCo bonds with finite maturity is presented in [6], whereas [1] also considers the case of unlimited maturity. Such a result is of practical importance since some of these products offered on the market have maturity equal to infinity (cf. [28]). In a situation where the CoCo bond has infinite maturity and the coupons of the non-optional part are floating, it is then natural to ask for an instrument which allows to hedge the interest rate risk involved in the non-optional part of the contract. A fixed to floating interest rate swap with infinitely many exchanges could serve as a hedging product for the interest rate risk beared by CoCo bonds. The main input for defining such a swap is its fixed rate, i.e. the long-term swap rate. Furthermore the long-term swap rate may also play an important role in the context of multiple curve bootstrapping. As we shall see in the following, we will concentrate our investigations on overnight indexed swap (OIS) contracts.

Such OIS contracts constitute the input quotes for bootstrapping procedures which allow for the construction of a discounting curve, according to the post-crisis market practice (cf. for example [9] or [22]). In view of this, the long-term swap rate becomes quite a natural object from which information on the long-end of the discounting curve can be inferred.

The main result of the paper is then the definition of the long-term swap rate and the study of its properties and relations with the long-term yield and the long-term simple rate. In particular we obtain that the long-term swap rate always exists finitely and that this rate is either constant or non-monotonic. In the case of a convergent infinite sum S_∞ of bonds we are able to provide an explicit model-free formula for R, which is only dependent on S_∞, on the time distance between the exchange dates that we consider as fixed, and on the bond price with maturity equal to the first reset date of the swap, see (4.1). Hence the long-term swap rate could represent an alternative discounting tool for long-term investments, since it is always finite, non-monotonic, can be explicitly characterised, and can be inferred by products existing on the markets.

As a contribution to the ongoing discussion on suitable discounting factors for investments over long time horizons, we then provide a comprehensive analysis of the relations among the long-term yield, the long-term simple rate and the long-term swap rate in a model-free approach. In particular we study how the existence of one of these long-term rates impacts the existence and finiteness of the other ones. This analysis shows the advantage of using the long-term swap rate as discounting rate, since it always remains finite when the other rates may become zero or explode.

The paper is structured as follows. First, we introduce in Section 2 some necessary prerequisites such as the different kinds of interest rates and interest rate swaps, in particular OISs. Then, Sections 3 and 4 describe the three asymptotic rates and some important features of the long-term swap rate like the model-free formula. In Section 5 we investigate the influence of each long-term rate on the existence and finiteness of the other rates. Finally, in Section 6 we analyse the long-term rates in some selected term structure models. We chose the Flesaker-Hughston methodology, developed in [18], and the linear-rational term structure methodology, presented in [15], because of some appealing features such as high tractability and simple forms of the different interest rates. In both cases we compute the long-term swap rate and the other long-term rates.

2 Fixed Income Setup

2.1 Interest Rates

We now introduce some notations. All quantities in the following are assumed to be associated to a risk-free curve, which, in the post-crisis market setting, can be approximated by the overnight curve used in collateralised transactions (cf. Section 1.1 of [9]).

First, we define the contract value of a zero-coupon bond at time t with maturity $T > t$ as $P(t,T)$. It guarantees its holder the payment of one unit of currency at time T, hence $P(T,T) = 1$ for all $T \geq 0$. We assume that there exists a frictionless market for zero-coupon bonds for every time $T > 0$ and that $P(t,T)$ is differentiable in T. In the following we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis of right-continuity and completeness. Furthermore, for no-arbitrage reasons we only consider finite positive zero-coupon bond prices, i.e. $0 < P(t,T) < +\infty$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$. Then, we define the yield for $[t,T]$ as the continuously compounded spot rate for $[t,T]$

$$Y(t,T) := -\frac{\log P(t,T)}{T-t}. \tag{2.1}$$

The simple spot rate for $[t,T]$ is

$$L(t,T) := \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right). \tag{2.2}$$

The short rate at time t is defined as

$$r_t := \lim_{T\downarrow t} Y(t,T) \quad \mathbb{P}\text{-a.s.} \tag{2.3}$$

The corresponding money-market account is denoted by $(\beta_t)_{t\geq 0}$ with

$$\beta_t := \exp\left(\int_0^t r_s\, ds\right). \tag{2.4}$$

All processes defined in (2.1) - (2.4) as well as the zero-coupon bond price process are assumed to be adapted processes on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. In particular we assume an arbitrage-free setting, where the discounted bond price process $\frac{P(t,T)}{\beta_t}, t \in [0,T]$, is an $(\mathbb{F}, \mathbb{P})$-martingale for all $T > 0$. We assume to work with the càdlàg version of $\frac{P(t,T)}{\beta_t}, t \in [0,T]$, for all $T > 0$. Consequently $P(t,T)\,, Y(t,T)\,, L(t,T)\,, t \in [0,T]$, are all càdlàg processes in the sequel.

2.2 Interest Rate Swaps

Swap contracts are derivatives where two counterparties exchange cashflows. There exist different kinds of swap contracts, involving cashflows deriving for example from commodities, credit risk or loans in different currencies. As far as interest rate swaps are concerned, the evaluation of such claims represents an aspect which is part of the discussion on multiple curve models, due to the recent financial crisis. While a survey of the literature on multiple-curve models would be beyond the scope of the present paper[1], we limit ourselves to note that even in the post-crisis setting, there are particular types of interest rate swaps whose evaluation formulas are equivalent to the ones employed for standard interest rate swaps in the single-curve pre-crisis setting. Since such instruments, called OISs play a pivotal role in the construction of discount curves, we concentrate our study on them, and avoid to define a full multiple-curve model.

Let us introduce a tenor structure of the form

$$t \leq T_0 < T_1 < \cdots < T_n = T\,, \tag{2.5}$$

where, for the sake of simplicity, we assume $\delta := T_i - T_{i-1}$ to be a constant for all $i = 1, \dots, n$. In an OIS contract, floating payments are indexed to a compounded overnight rate like EONIA. The variable rate that one party has to pay every time T_i, $i = 1, \dots, N$, is $\delta \bar{L}(T_{i-1}, T_i)$ with $\bar{L}(T_{i-1}, T_i)$ denoting the compounded overnight rate for $[T_{i-1}, T_i]$. This rate is given by

$$\bar{L}(T_{i-1}, T_i) = \frac{1}{\delta}\left(\exp\left(\int_{T_{i-1}}^{T_i} r_s\, ds\right) - 1\right).$$

Then, the OIS rate, i.e. the fixed rate which makes the OIS value equal to zero at inception, is

$$\begin{aligned} R_\delta^{OIS}(t,T) &= \frac{\sum_{i=1}^n \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^{T_i} r_s\, ds\right)\delta \bar{L}(T_{i-1}, T_i)\Big|\, \mathcal{F}_t\right]}{\delta \sum_{i=1}^n P(t, T_i)} \\ &= \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^n P(t,T_i)}\,, \end{aligned}$$

which corresponds to the formula for the par swap rate in a single curve setting. In the following, we consider only OIS swaps, hence we set

$$R(t,T) := R_\delta^{OIS}(t,T) \tag{2.6}$$

for all $t, T \geq 0$.

[1] For a complete list of references the interested reader is referred to [9].

3 Long-Term Rates

In this section we consider some possible long-term rates. In particular we focus on the long-term yield and the long-term simple rate, which have been already defined in the literature (cf. [25] and [7]). The *long-term yield* can be defined in different ways. Some articles investigate interest rates with a certain time to maturity to approach the concept of "long-term", e.g. in [33] yield curves with time to maturity over 30 years are examined, [32] considers yields with a maturity beyond 20 years to be "long-term", whereas the ECB takes 10 years as a barrier, cf. [13]. Another approach is to look at the asymptotic behaviour of the yield curve by letting the maturity go to infinity. This approach is used by [3], [4], [12], [25]. In line with the above-mentioned principle, we introduce our first object of study, and define the long-term yield $\ell := (\ell_t)_{t\geq 0}$ as

$$\ell_\cdot := \lim_{T\to\infty} Y(\,\cdot\,, T)\,, \tag{3.1}$$

if the limit exists in the sense of the uniform convergence on compacts in probability (convergence in ucp).[2] If the limit in (3.1) exists but it is infinite, positive or negative, see Definition B.3, we will write $\ell = \pm\infty$ for the sake of simplicity. We will use this improper notation also for the other long-term interest rates in the sequel of the paper. We recall that the long-term yield process ℓ is a non-decreasing process by the DIR theorem, which was first proved in [12] and further discussed in [20], [23], and [24].

In [7] it is suggested to consider a particular model for the *long-term simple rate* for the discounting of cashflows occuring in a distant future. By using exponential discount factors the discounted value of a long-term project, that will be realised over a long time horizon, in most cases will turn out to be overdiscounted, hence too small to justify the overall project costs. To overcome this problem, the authors of [7] came up with the concept of "social discounting", where the long-term simple rate is employed for discounting cashflows in the distant future. To integrate this interesting approach into our considerations, we now define the long-term simple rate process $L := (L_t)_{t\geq 0}$ as

$$L_\cdot := \lim_{T\to\infty} L(\,\cdot\,, T)\,,$$

if the limit exists in ucp, where $L(t,T)$ is defined in (2.2). Note that $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ by (2.2).

[2] For a definition of the ucp convergence and some additional results the reader is referred to Section B in the appendix.

We define the *long-term bond* $P := (P_t)_{t\geq 0}$ as

$$P_{\cdot} := \lim_{T\to\infty} P(\cdot, T)\,, \tag{3.2}$$

if the limit exists in ucp.

Remark 3.1. *We note that as a consequence of our assumption that the bond prices are càdlàg, we also obtain that all the long-term rates introduced above and in Section 4 are càdlàg. In the sequel we will then use Theorem 2 of Chapter I, Section 1 of [29], which tells us that for two right-continuous stochastic processes X and Y it holds that $X_t = Y_t$ $\mathbb{P}$-a.s. for all $t \geq 0$ is equivalent to $\mathbb{P}$-a.s. for all $t \geq 0$, $X_t = Y_t$.*

We define $S_n := (S_n(t))_{t\geq 0}$ with

$$S_n(t) := \sum_{i=1}^{n} P(t, T_i)\,, t \geq 0,$$

considering a tenor structure with infinite many exchange dates. Then the limit in ucp

$$S_\infty(\cdot) := \lim_{n\to\infty} S_n(\cdot) = \lim_{n\to\infty} \sum_{i=1}^{n} P(\cdot, T_i)\,, \tag{3.3}$$

always exists, finite or infinite. All bond prices are strictly positive, therefore for all $t \geq 0, n \in \mathbb{N}$ we have $\mathbb{P}$-a.s. $S_n(t) > 0$ and $S_\infty(t) > 0$.

4 The Long-Term Swap Rate

We now introduce the *long-term swap rate* $R := (R_t)_{t\geq 0}$ as

$$R_{\cdot} := \lim_{n\to\infty} R(\,\cdot\,, T_n) = \lim_{T\to\infty} R(\,\cdot\,, T)\,,$$

if the limit exists in ucp, where $R(t, T)$ is defined in (2.6). The long-term swap rate, defined here for the first time, can be understood as the fair fixed rate of an OIS that has a payment stream with infinitely many exchanges. This fixed rate is meant to be fair in the sense that the initial value of this OIS equals zero.

We investigate the existence and finiteness of the long-term swap rate. We first provide a model-free formula for the swap rate, when S_∞ exists and is finite.

Proposition 4.1. *If* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ *in ucp, then* $\mathbb{P}$*-a.s.*

$$R_t = \frac{P(t,T_0)}{\delta S_\infty(t)} > 0 \tag{4.1}$$

for all $t \geq 0$*.*

Proof. Considering the tenor structure (2.5) we have that in ucp

$$\begin{aligned}
\lim_{n\to\infty} R(\,\cdot\,,T_n) &\overset{(2.6)}{=} \lim_{n\to\infty} \frac{P(\,\cdot\,,T_0) - P(\,\cdot\,,T_n)}{\delta S_n(\,\cdot\,)} \\
&= \lim_{n\to\infty} \frac{P(\,\cdot\,,T_0)}{\delta S_n(\,\cdot\,)} - \lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{\delta S_n(\,\cdot\,)} = \lim_{n\to\infty} \frac{P(\,\cdot\,,T_0)}{\delta S_n(\,\cdot\,)} \\
&= \frac{P(\,\cdot\,,T_0)}{\delta} \lim_{n\to\infty} \frac{1}{S_n(\cdot)} = \frac{P(\,\cdot\,,T_0)}{\delta S_\infty(\,\cdot\,)} > 0
\end{aligned}$$

by Theorem B.2. □

Proposition 4.2. *If* $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp and the long-term bond* P*, defined in* (3.2)*, exists finitely, then it holds* $R_t = 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*.*

Proof. Considering the tenor structure (2.5) we have that in ucp

$$\begin{aligned}
\lim_{n\to\infty} R(\,\cdot\,,T_n) &\overset{(2.6)}{=} \lim_{n\to\infty} \frac{P(\,\cdot\,,T_0) - P(\,\cdot\,,T_n)}{\delta S_n(\,\cdot\,)} \\
&= \lim_{n\to\infty} \frac{P(\,\cdot\,,T_0)}{\delta S_n(\,\cdot\,)} - \lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{\delta S_n(\,\cdot\,)} \\
&= \frac{P(\,\cdot\,,T_0)}{\delta} \lim_{n\to\infty} \frac{1}{S_n(\cdot)} - \frac{1}{\delta} \lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} \\
&= -\frac{1}{\delta} \lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} = -\frac{P_\cdot}{\delta} \lim_{n\to\infty} \frac{1}{S_n(\cdot)} = 0
\end{aligned}$$

by Theorem B.2. □

By Propositions 4.1 and 4.2 we obtain the existence of the long-term swap rate as a finite limit if the long-term bond price exists finitely. However this result always holds as shown by the following corollary.

Corollary 4.3. *The long-term swap rate cannot explode, i.e.* $\mathbb{P}(|R_t| < +\infty) = 1$ *for all* $t \geq 0$*.*

Proof. Simply note that $\mathbb{P}$-a.s.

$$0 \leq \sup_{0\leq s\leq t} \frac{P(s,T_n)}{S_n(s)} = \sup_{0\leq s\leq t} \left(1 - \frac{S_{n-1}(s)}{S_n(s)}\right) \leq 1$$

for all $t \geq 0$. Hence

$$\mathbb{P}\left(\inf_{0\leq s\leq t} \frac{P\left(s,T_n\right)}{S_n(s)} > M\right) \overset{n\to\infty}{\longrightarrow} 0$$

for all $M > 1$. $\square$

Proposition 4.4. *Suppose $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. If for all $t \geq 0$ it holds $r_t \geq 0$ $\mathbb{P}$-a.s., then*

$$R_t = -\frac{k_t}{\delta}$$

for a process $(k_t)_{t\geq 0}$ with $0 \leq k_t \leq 1$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Proof. If $r_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, we have for the tenor structure (2.5) that

$$\begin{aligned}\frac{P(t,T_n)}{\beta_t} &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^{T_n} r_s\,ds\right)\middle|\mathcal{F}_t\right]\\ &\geq \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^{T_{n+1}} r_s\,ds\right)\middle|\mathcal{F}_t\right] = \frac{P(t,T_{n+1})}{\beta_t}.\end{aligned}$$

Since for all $n \in \mathbb{N}$, $S_n(t) \leq S_{n+1}(t)$ $\mathbb{P}$-a.s. for all $t \geq 0$, for all $n \in \mathbb{N}$ we have $\mathbb{P}$-a.s.

$$\frac{P(t,T_n)}{S_n(t)} = \frac{P\left(t,T_n\right)}{\beta_t}\frac{\beta_t}{S_n(t)} \geq \frac{P\left(t,T_{n+1}\right)}{\beta_t}\frac{\beta_t}{S_{n+1}(t)} = \frac{P(t,T_{n+1})}{S_{n+1}(t)}$$

for all $t \geq 0$. This implies that $\mathbb{P}$-a.s.

$$1 \geq \sup_{0\leq s\leq t} \frac{P(s,T_n)}{S_n(s)} \geq \sup_{0\leq s\leq t} \frac{P(s,T_{n+1})}{S_{n+1}(s)}$$

for all $t \geq 0$. Hence $\frac{P(\,\cdot\,,T_n)}{S_n(\cdot)} \overset{n\to\infty}{\longrightarrow} k.$ in ucp, with $0 \leq k_t \leq 1$ $\mathbb{P}$-a.s. for all $t \geq 0$. In particular we get $k_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if $P_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ by Proposition 4.2. $\square$

If we assume that there exists a liquid market for perpetual OIS, meaning OIS with infinitely many exchanges with the fixed rate corresponding to the long-term swap rate, we can state the following theorem. We recall that we are working under the hypothesis that $\mathbb{P}$ is an equivalent martingale measure for the bond market, i.e that the bond market is arbitrage-free in the sense of Section 4.3 of [14].

Theorem 4.5. *In the setting outlined in Section 2.1 the long-term swap rate is either constant or non-monotonic.*

Proof. First, we assume that $R_s \geq R_t$ $\mathbb{P}$-a.s. with $\mathbb{P}(R_s > R_t) > 0$ for $0 \leq t \leq s \leq T_0$. Then, let us consider the following investment strategy. At time t we enter a payer OIS with perpetual annuity, nominal value N, fixed-rate R_t and the following tenor structure

$$t \leq T_0 < T_1 < \cdots < T_n \tag{4.2}$$

where $n \to \infty$. This investment has zero value in t, so there is no net investment so far. We receive the following payoff in each T_i, $i \in \mathbb{N}$:

$$\big(\bar{L}(T_{i-1}, T_i) - R_t\big)\, \delta N\,.$$

Then at time s we enter a receiver OIS with a perpetual annuity, nominal value N, a fixed-rate of R_s and the same tenor structure as in (4.2). The value of this OIS is zero in s, hence there is still no net investment, and the payoff in each T_i, $i \in \mathbb{N}$, resulting from this OIS is:

$$\big(R_s - \bar{L}(T_{i-1}, T_i)\big)\, \delta N\,.$$

This strategy leads to the payoff

$$H_i := \big(\bar{L}(T_{i-1}, T_i) - R_t\big)\, \delta N + \big(R_s - \bar{L}(T_{i-1}, T_i)\big)\, \delta N = \delta N\,(R_s - R_t) \geq 0$$

with $\mathbb{P}(H_i > 0) > 0$, i.e. to an arbitrage.

If we assume that $R_s \leq R_t$ $\mathbb{P}$-a.s. with $\mathbb{P}(R_s < R_t) > 0$ for $0 \leq t \leq s \leq T_0$, we use an analogue arbitrage strategy with the only difference that we invest in t in a receiver OIS and in s in a payer OIS.

It follows that in an arbitrage-free market setting the long-term swap rate cannot be non-decreasing or non-increasing, i.e. it can only be monotonic if it is constant. $\square$

5 Relation between Long-Term Rates

We now study the relation among the long-term rates introduced in Sections 3 and 4 in terms of their existence. For further details, we also refer to [21].

5.1 Influence of the Long-Term Yield on Long-Term Rates

In this section we study the influence of the existence of the long-term yield on the existence of the long-term swap and simple rates. Since typical market data indicate positive long-term yields[3], we restrict ourselves to the cases of $\ell \geq 0$. For a more general analysis which also takes into account the possibility of a negative long-term yield, we refer to [21].

Theorem 5.1. *If $0 < \ell_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $L = +\infty$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. For this, it is sufficient to show that for all $t \geq 0$, $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < +\infty$ $\mathbb{P}$-a.s., since this also implies $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < +\infty$ in probability.

We know that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |Y(s,T_n) - \ell_s| \leq \epsilon\right) \overset{(2.1)}{=} \mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{\log P(s,T_n)}{T_n - s} + \ell_s\right| \leq \epsilon\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. for all $t \geq 0$ and all $\epsilon > 0$ there exists $N^t_\epsilon \in \mathbb{N}$ such that for all $n \geq N^t_\epsilon$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{\log P(s,T_n)}{T_n - s} + \ell_s\right| \leq \epsilon\right) > 1 - \delta(\epsilon) \tag{5.1}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_1^{\epsilon,u,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq u} \left|\frac{\log P(s,T_n)}{T_n - s} + \ell_s\right| \leq \epsilon\right\}. \tag{5.2}$$

Then for $n \geq N^u_\epsilon$ with $u > t$ we have $\mathbb{P}(A_1^{\epsilon,u,n}) > 1 - \delta(\epsilon)$ by (5.1) and

$$A_1^{\epsilon,u,n} \subseteq \{\omega \in \Omega : |\log P(t,T_n) + (T_n - t)\,\ell_t| \leq \epsilon\,(T_n - t)\}.$$

Consequently for $n \geq N^u_\epsilon$ on $A_1^{\epsilon,u,n}$ we have

$$\exp[-(\epsilon + \ell_t)\,(T_n - t)] \leq P(t,T_n) \leq \exp[(\epsilon - \ell_t)\,(T_n - t)] \tag{5.3}$$

for all $t \in [0,u]$ and since $\ell_0 \leq \ell_t \leq \ell_u$ for all $t \in [0,u]$ by the DIR theorem, we have that for $n \geq N^u_\epsilon$ on $A_1^{\epsilon,u,n}$ it holds

$$\exp[-(\epsilon + \ell_u)\,(T_n - u)] \leq \sup_{0\leq s\leq t} P(s,T_n) \leq \exp[(\epsilon - \ell_0)\,T_n]. \tag{5.4}$$

[3]For long-term interest rate market data please refer to [13] for the EUR market and to [5] for the USD market.

For $t \geq 0$ we define

$$B_1(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < +\infty \right\}. \tag{5.5}$$

We then obtain for $t < u$ and $n \geq N_\epsilon^u$

$$\begin{aligned}
\mathbb{P}(B_1(t)) &= \mathbb{P}\left(\left\{ \sup_{0\leq s\leq t} S_{N_\epsilon^u - 1}(s) < +\infty \right\} \cap \left\{ \lim_{n\to+\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \right\} \right) \\
&= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \right) \\
&= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\quad + \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \,\middle|\, \Omega\backslash A_1^{\epsilon,u,n} \right) \mathbb{P}(\Omega\backslash A_1^{\epsilon,u,n}) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0\leq s\leq t} P(s,T_i) < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\overset{(5.4)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \exp[(\epsilon-\ell_0)\,T_i] < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\geq (1-\delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to 0$ since it holds $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{\exp\left(-\ell_0 T_{n+1}\right)}{\exp\left(-\ell_0 T_n\right)} = \exp(-\ell_0\delta) \in (0,1)\,,$$

which implies by the ratio test that $\lim_{n\to\infty} \sum_{i=0}^{n} \exp[(\epsilon-\ell_0)\,T_i] < +\infty$ $\mathbb{P}$-a.s. for $\epsilon \to 0$. That means, it holds $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.

Hence by Proposition 4.1 and Corollary 4.3 we get for all $t \geq 0$ that $0 < R_t < +\infty$ $\mathbb{P}$-a.s. with

$$R_t = \frac{P(t,T_0)}{\delta S_\infty(t)}.$$

The exploding long-term simple rate, $L = +\infty$, is a result of Remark 3 of [7]. $\square$

We were able to derive the important result that a finite positive long-term yield implies a finite positive long-term swap rate.

Now, let us investigate what happens to the long-term rates if the long-term yield either vanishes or explodes. We see that besides the asymptotic behaviour of the yield, information about the long-term zero-coupon bond price is needed to state the consequences on the other long-term rates.

Proposition 5.2. *Let $\ell_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. If the long-term bond price P exists finitely with* $\inf_{0\leq s\leq t} P_s > 0$ *$\mathbb{P}$-a.s. for all $t \geq 0$, then $R_t = 0$ and $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. From Corollary A.2 follows that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp, hence by applying Proposition 4.2 we get that $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

To show that the long-term simple rate vanishes $\mathbb{P}$-a.s., we prove that for all $t \geq 0$ it holds that $\mathbb{P}(B_2(t)) = 1$ with $B_2(t)$ defined for $t \geq 0$ as follows

$$B_2(t) := \left\{\omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} L(s, T_n) = 0\right\}.$$

We have for all $t \geq 0$

$$\begin{aligned}
\mathbb{P}(B_2(t)) &\overset{(2.2)}{=} \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \frac{1}{(T_n - s)\, P(s, T_n)} = 0\right) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \frac{1}{(T_n - t)\inf_{0\leq s\leq t} P_s} = 0\right) = 1.
\end{aligned}$$

□

In the following, we investigate exploding long-term yields.

Theorem 5.3. *If $\ell = +\infty$, then $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $L = +\infty$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. We know by (B.6) that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\inf_{0\leq s\leq t} |Y(s, T_n)| > \epsilon\right) \overset{(2.1)}{\geq} \mathbb{P}\left(\inf_{0\leq s\leq t} |\log P(s, T_n)| > \epsilon\, T_n\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. for all $t \geq 0$ and all $\epsilon > 0$ there exists a $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(\inf_{0\leq s\leq t} |\log P(s, T_n)| > \epsilon\, T_n\right) > 1 - \delta(\epsilon) \tag{5.6}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_2^{\epsilon,u,n} := \left\{ \omega \in \Omega : \inf_{0 \leq s \leq u} |\log P(s, T_n)| > \epsilon T_n \right\}. \tag{5.7}$$

Then for $n \geq N_\epsilon^u$, $t < u$ and $B_1(t)$ defined as in (5.5), we obtain

$$\begin{aligned}
\mathbb{P}(B_1(t)) &= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s, T_i) < +\infty \right) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0\leq s\leq t} P(s, T_i) < +\infty \middle| A_2^{\epsilon,u,n} \right) \mathbb{P}(A_2^{\epsilon,u,n}) \\
&\overset{(5.7)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \exp(-\epsilon T_n) < +\infty \middle| A_2^{\epsilon,u,n} \right) \mathbb{P}(A_2^{\epsilon,u,n}) \\
&\geq (1 - \delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to 0$ due to the ratio test. That means $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp and consequently $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Proposition 4.1.

Remark 3 of [7] leads to $L = +\infty$. □

The following table summarises the influence of the long-term yield on the long-term swap rate and long-term simple rate.

If the long-term yield is	With long-term bond price	Then the long-term swap rate is	Then the long-term simple rate is
$\ell = 0$	$0 < P < +\infty$	$R = 0$	$L = 0$
$\ell = 0$	$0 \leq P \leq +\infty$	$0 \leq R < +\infty$	$0 \leq L \leq +\infty$
$\ell > 0$	$0 \leq P \leq +\infty$	$0 < R < +\infty$	$L = +\infty$

Table 1 – Influence of the long-term yield on long-term rates.

5.2 Influence of the Long-Term Swap Rate on Long-Term Rates

After we investigated the influence of the long-term yield on the long-term swap rate and long-term simple rate, we are also interested in the other direction of this relation. First, we look at the consequences of a vanishing long-term swap rate.

Theorem 5.4. *If $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. For this, let us assume S_n converges in ucp. Then, according to Proposition 4.1 it is $0 < R_t$ $\mathbb{P}$-a.s. for all $t \geq 0$, but this is a contradiction and therefore S_n converges to $+\infty$ in ucp.

Consequently $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Theorems 5.1 and 5.3. $\square$

Now, we investigate the behaviour of the long-term rates if the long-term swap rate is strictly positive.

Theorem 5.5. *If $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then it holds:*

(i) If the long-term bond price P exists finitely, then $\ell_t \geq 0$ and $L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

(ii) If the long-term bond price P exists infinitely, then $\ell_t \geq 0$ and $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Proof. First, we investigate the behaviour of the infinite bond sum. We know from Proposition 4.2 that $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if S_n converges to $+\infty$ in ucp and P exists finitely. Hence if $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, we have $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp or $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp with $P = +\infty$.
To (i): In this case it holds $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. Then, according to Propositions 3.2.3 and 3.2.9 of [21] it holds $\mathbb{P}$-a.s. $\ell_t \geq 0$ for all $t \geq 0$.

Further, $L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ is a consequence of Proposition 3.2.11 of [21].
To (ii): In this case it holds $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp due to Corollary A.2. Then, according to Theorems 5.1 and 5.3 it holds $\mathbb{P}$-a.s. $\ell_t \leq 0$ for all $t \geq 0$.

Further, from Lemma A.3 it follows that $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. $\square$

Theorem 5.5 shows that in order to analyse the relation between long-term swap rate and the other long-term rates we need to specify the behaviour of the long-term bond.

The only case left now is a strictly negative long-term swap rate.

Theorem 5.6. *If $-\infty < R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \leq 0$ and $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. We know from Proposition 4.1 that $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if S_n converges to S_∞ in ucp, but this is a contradiction to $R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

As a consequence of Theorems 5.1 and 5.3 it is $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Due to Proposition 4.2 we know that P does not exist finitely, hence $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ because of Lemma A.3. $\square$

In the table below we summarise the influence of the long-term swap rate on the other long-term rates. Note, that $-\infty < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ by Proposition 4.3. Hence, only three different cases have to be distinguished, $R_t = 0$, $0 < R_t < +\infty$, and $-\infty < R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

If the long-term swap rate is	With long-term bond price	Then the long-term yield is	Then the long-term simple rate is
$R = 0$	$0 \leq P < +\infty$	$\ell \leq 0$	$0 \leq L \leq +\infty$
$R = 0$	$P = +\infty$	$\ell \leq 0$	$L = 0$
$0 < R < +\infty$	$0 \leq P < +\infty$	$\ell \geq 0$	$0 < L \leq +\infty$
$0 < R < +\infty$	$P = +\infty$	$\ell \leq 0$	$L = 0$
$-\infty < R < 0$	$0 \leq P \leq +\infty$	$\ell \leq 0$	$L = 0$

Table 2 – Influence of the long-term swap rate on long-term rates.

5.3 Influence of the Long-Term Simple Rate on Long-Term Rates

Finally, we want to know about the influence of the long-term simple rate on long-term yields and long-term swap rates. Since $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, it is sufficient to investigate the three different cases, where $L_t = 0$, or $0 < L_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, or $L = +\infty$. First, we look at a vanishing long-term simple rate.

Theorem 5.7. *If $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Furthermore, $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if $P_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. We know that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |L(s, T_n)| \leq \epsilon\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. by (2.2) for all $t \geq 0$ and all $\epsilon > 0$ there exists $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{1}{T_n - s}\left(\frac{1}{P(s, T_n)} - 1\right)\right| \leq \epsilon\right) > 1 - \delta(\epsilon) \tag{5.8}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_3^{\epsilon,u,n} := \left\{ \omega \in \Omega : \sup_{0 \leq s \leq u} \left| \frac{1}{T_n - s} \left(\frac{1}{P(s,T_n)} - 1 \right) \right| \leq \epsilon \right\}. \tag{5.9}$$

Let us define for $t \geq 0$

$$B_3(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \inf_{0\leq s\leq t} S_n(s) = +\infty \right\}.$$

For $t < u$ and $n \geq N_\epsilon^u$ we then obtain

$$\begin{aligned} \mathbb{P}(B_3(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \inf_{0\leq s\leq t} P(s,T_i) = +\infty \middle| A_3^{\epsilon,u,n} \right) \mathbb{P}(A_3^{\epsilon,u,n}) \\ &\overset{(5.9)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \frac{1}{1+\epsilon T_i} = +\infty \middle| A_3^{\epsilon,u,n} \right) \mathbb{P}(A_3^{\epsilon,u,n}) \\ &\geq (1-\delta(\epsilon)) \to 1 \end{aligned}$$

for $\epsilon \to 0$. That means $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp and consequently $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Theorems 5.1 and 5.3.

The vanishing of the long-term swap rate is a direct consequence of Proposition 4.2. □

If the long-term bond price explodes, it is not possible to specify the long-term swap rate more accurately than in Proposition 4.4.

Next, we investigate the case when the long term simple rate is strictly positive.

Theorem 5.8. *If* $0 < L_t < +\infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*, then* $\ell_t \geq 0$ *and* $R_t > 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. We know that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |L(s,T_n) - L_s| \leq \epsilon \right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. by (2.2) for all $t \geq 0$ and all $\epsilon > 0$ there exists $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \left| \frac{1}{T_n - s} \left(\frac{1}{P(s,T_n)} - 1 \right) - L_s \right| \leq \epsilon \right) > 1 - \delta(\epsilon)$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_4^{\epsilon,u,n} := \left\{ \omega \in \Omega : \sup_{0 \leq s \leq u} \left| \frac{1}{T_n - s} \left(\frac{1}{P(s, T_n)} - 1 \right) - L_s \right| \leq \epsilon \right\}. \tag{5.10}$$

Let $B_1(t)$ defined as in (5.5). For $t < u$ and $n \geq N_\epsilon^u$ we then obtain

$$\begin{aligned}
\mathbb{P}(B_1(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0\leq s\leq t} P(s,T_i) < +\infty \,\middle|\, A_4^{\epsilon,u,n} \right) \mathbb{P}(A_4^{\epsilon,u,n}) \\
&\overset{(5.10)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \frac{1}{L_t^* \, (T_i - t) + 1 - \epsilon \, T_i} < +\infty \,\middle|\, A_4^{\epsilon,u,n} \right) \mathbb{P}(A_4^{\epsilon,u,n}) \\
&\geq (1 - \delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to 0$, where $L_t^* := \inf_{0\leq s\leq t} L_s$. That means $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp and by Proposition A.1 it holds $P_t = 0$ for all $t \geq 0$. Then, by Proposition 3.2.3 and 3.2.9 of [21] it follows that $\ell_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Moreover, $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ as a consequence of Proposition 4.1. □

Lastly, we are interested in the influence of an exploding long-term simple rate on the long-term yield and long-term swap rate.

Theorem 5.9. *If $L = +\infty$, then $\ell_t \geq 0$ and $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. We show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, then the rest of the proof is exactly as in the proof of Theorem 5.8. We know by (B.6) that for all $t \geq 0$ and all $\epsilon > 0$

$$\mathbb{P}\left(\inf_{0\leq s\leq t} L(s, T_n) > \epsilon \right) \overset{n\to\infty}{\longrightarrow} 1.$$

Hence it holds by (2.2) for all $t \geq 0$ and all $\epsilon > 0$ that there exists $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(T_n \sup_{0\leq s\leq t} P(s, T_n) \leq \epsilon \right) > 1 - \delta(\epsilon) \tag{5.11}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Then, let us define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_5^{\epsilon,u,n} := \left\{ \omega \in \Omega : T_n \sup_{0\leq s\leq u} P(s, T_n) \leq \epsilon \right\}. \tag{5.12}$$

For $t < u$ and $n \geq N_\epsilon^u$ it holds with $B_1(t)$ defined as in (5.5)

$$\begin{aligned}
\mathbb{P}(B_1(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0\leq s\leq t} P(s,T_i) < +\infty \,\middle|\, A_5^{\epsilon,u,n} \right) \mathbb{P}(A_5^{\epsilon,u,n}) \\
&\overset{(5.12)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \frac{1}{T_i}\epsilon < +\infty \,\middle|\, A_5^{\epsilon,u,n} \right) \mathbb{P}(A_5^{\epsilon,u,n}) \\
&\geq (1-\delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to 0$. □

Table 3 summarises the influence of the long-term simple rate on the other long-term rates.

If the long-term simple rate is	With long-term bond price	Then the long-term yield is	Then the long-term swap rate is
$L = 0$	$0 \leq P < +\infty$	$\ell \leq 0$	$R = 0$
$L = 0$	$P = +\infty$	$\ell \leq 0$	$-\infty < R < +\infty$
$0 < L < +\infty$	$P = 0$	$\ell \geq 0$	$0 < R < +\infty$
$L = +\infty$	$P = 0$	$\ell \geq 0$	$0 < R < +\infty$

Table 3 – Influence of the long-term simple rate on long-term rates.

6 Long-Term Rates in Specific Term Structure Models

In this section we compute the long-term interest rates in two specific models, the Flesaker-Hughston model and the linear-rational model. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the filtered probability space introduced in Section 2.1.

6.1 Long-Term Rates in the Flesaker-Hughston Model

We now derive the long-term swap rate in the Flesaker-Hughston interest rate model. The model has been introduced in [18] and further developed in [27] and [30]. Main advantages of this approach are that it specifies non-negative interest rates only and has a high degree of tractability. Another appealing feature is that besides relatively simple models for bond prices, short and forward rates, there are closed-form formulas for caps, floors and swaptions available. In the following, we first shortly outline the generalised Flesaker-Hughston model that is explained in detail in [30] and then consider two specific cases. The basic input of the model is a strictly positive supermartingale A on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ which represents the state price density, so that the zero-coupon bond price can be expressed as

$$P(t,T) = \frac{\mathbb{E}^{\mathbb{P}}[\,A_T | \,\mathcal{F}_t]}{A_t}\,,\ 0 \leq t \leq T\,, \tag{6.1}$$

for all $T \geq 0$. It immediately follows $P(T,T) = 1$ for all $T \geq 0$ and $P(t,U) \leq P(t,T)$ for all $0 \leq t \leq T \leq U$, i.e. the zero-coupon bond price is a decreasing process in the maturity. This choice guarantees positive forward and short rates for all maturities (cf. equations (10) and (11) of [18]). To model the long-term yield and swap rate in this methodology a specific choice of A is needed. For this matter, we focus on two special cases presented in Section 2.3 of [30].

Example 6.1. *The supermartingale A is given by*

$$A_t = f(t) + g(t)\, M_t\,,\ t \geq 0\,,$$

where $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly positive decreasing functions and M is a strictly positive martingale defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $M_0 = 1$. We shall consider in the sequel a càdlàg version of M. It follows from (6.1) *that for all $0 \leq t \leq T$*

$$P(t,T) = \frac{f(T) + g(T)\, M_t}{f(t) + g(t)\, M_t}\,. \tag{6.2}$$

The initial yield curve can easily be fitted by choosing strictly positive decreasing functions f and g in such a way that

$$P(0,T) = \frac{f(T) + g(T)}{f(0) + g(0)}$$

for all $T \geq 0$.

For the calculations of the long-term yield and swap rate, we assume that the following conditions on the asymptotic behaviour of f and g hold:

$$F := \sum_{i=1}^{\infty} f(T_i) < +\infty\,, \qquad G := \sum_{i=1}^{\infty} g(T_i) < +\infty\,, \tag{6.3}$$

with $F, G \in \mathbb{R}_+$.

From (6.3) *it follows immediately that* $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0$ *and hence* $P_t = 0$ *for all* $t \geq 0$. *This condition is assumed in [18], whereas here it follows from* (6.3). *We also get for all* $t \geq 0$

$$S_\infty(t) = \frac{F + GM_t}{f(t) + g(t)\, M_t} \quad \mathbb{P}\text{-}a.s.$$

since

$$\begin{aligned}
&\sup_{0\leq s\leq t} \left| \sum_{i=1}^{n} \frac{f(T_i) + g(T_i)\, M_s}{f(s) + g(s)\, M_s} - \frac{F + GM_s}{f(s) + g(s)\, M_s} \right| \\
&\quad = \sup_{0\leq s\leq t} \left| \frac{M_s}{f(s) + g(s)\, M_s} \left(\sum_{i=1}^{n} g(T_i) - G \right) + \frac{\sum_{i=1}^{n} f(T_i) - F}{f(s) + g(s)\, M_s} \right| \\
&\quad \to 0 \quad \mathbb{P}\text{-}a.s.
\end{aligned}$$

for all $t \geq 0$, *hence in probability because*

$$\sup_{0\leq s\leq t} \frac{M_s}{f(s) + g(s)\, M_s} \leq \sup_{0\leq s\leq t} \frac{M_s}{g(s)\, M_s} \leq \frac{1}{g(t)} < +\infty.$$

Then, by Proposition 4.1 it holds $\mathbb{P}$-*a.s.*

$$R_t = \frac{f(T_0) + g(T_0)\, M_t}{\delta\,(F + GM_t)}, \ t \geq 0. \tag{6.4}$$

Now, we also want to compute the long-term yield in this model specification. It is for all $t \geq 0$

$$\ell_{\cdot} \overset{(3.1)}{=} \lim_{T\to\infty} Y(\,\cdot\,, T) \underset{(6.2)}{\overset{(2.1)}{=}} - \lim_{T\to\infty} T^{-1} \log(f(T) + g(T)\, M_{\cdot}) \quad in\ ucp. \tag{6.5}$$

We know from Theorem 5.5 that $\ell_t \geq 0$ $\mathbb{P}$-*a.s. for all* $t \geq 0$ *since the long-term swap rate is strictly positive due to* (6.4) *and the long-term bond price vanishes.*

Let us consider a simple example, where $f(t) = \exp(-\alpha t)$, $g(t) = \exp(-\beta t)$ *with* $0 < \alpha < \beta$. *Then* f *and* g *are decreasing strictly positive functions and the ratio test shows that the infinite sums of* f *and* g *exist. We denote them by*

$$\alpha_\infty := \sum_{i=1}^{\infty} \exp(-\alpha T_i)\ , \qquad \beta_\infty := \sum_{i=1}^{\infty} \exp(-\beta T_i)$$

with $0 < \beta_\infty \leq \alpha_\infty$. *Hence all required conditions are fulfilled and we get the following equations for the long-term swap rate and the long-term yield, respectively*

$$R_t \overset{(6.4)}{=} \frac{\exp(-\alpha T_0) + \exp(-\beta T_0)\, M_t}{\delta\, (\alpha_\infty + \beta_\infty M_t)}\,,\ t \geq 0,$$

and

$$\begin{aligned} \ell_. &\overset{(6.5)}{=} - \lim_{T\to\infty} T^{-1} \log(f(T) + g(T)\, M_.) \\ &= - \lim_{T\to\infty} T^{-1} \log(\exp(-\alpha T)\,(1 + \exp\left(-\left(\beta - \alpha\right) T\right) M_.)) \\ &= \alpha + \lim_{T\to\infty} T^{-1} \log(1 + \exp(-\left(\beta-\alpha\right) T)\, M_.) = \alpha \quad \textit{in ucp.} \end{aligned}$$

It follows by Theorem 5.1 that $L(t, T_n) \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp. This result can also be obtained by direct computation since for all* $t \geq 0$

$$\sup_{0\leq s\leq t} \frac{\exp(-\alpha s) + \exp(-\beta s)\, M_s}{T \exp(-\alpha T) + T \exp(-\beta T)\, M_s} \overset{T\to\infty}{\longrightarrow} +\infty \quad \mathbb{P}\textit{-a.s.},$$

i.e. in probability, since M *is càdlàg.*

Example 6.2. *In the second special case of the Flesaker-Hughston model the supermartingale* A *is defined as*

$$A_t = \int_t^\infty \phi(s)\, M(t,s)\; ds\,,\ t \geq 0\,,$$

where for every $s > 0$ *the process* $M(t,s)\,, t \leq s$*, is a strictly positive martingale on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *with* $M(0,s) = 1$ *such that* $\int_0^\infty \phi(s)\, M(t,s)\; ds < +\infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$ *and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strictly positive continuous function. From* (6.1) *follows for* $0 \leq t \leq T$

$$P(t,T) = \frac{\int_T^\infty \phi(s)\, M(t,s)\; ds}{\int_t^\infty \phi(s)\, M(t,s)\; ds}\,, \tag{6.6}$$

for all $T \geq 0$. *By differentiation of the zero-coupon bond price with respect to the maturity date, we see that the initial term structure satisfies* $\phi(t) = -\frac{\partial P(0,t)}{\partial t}$ *(cf. equation (6) of [18]).*

According to (6.6) *we get that* $P_t = 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$. *We define* $Q_n := (Q_n(t))_{t\geq 0}$ *for all* $n \geq 0$ *with*

$$Q_n(t) := \sum_{i=1}^n \int_{T_i}^\infty \phi(s)\, M(t,s)\; ds$$

and assume that for $Q := (Q(t))_{t\geq 0}$ *we have*

$$Q(t) := \sum_{i=1}^{\infty}\int_{T_i}^{\infty} \phi(s)\, M(t,s)\; ds \, < +\infty$$

for all $t \geq 0$*, and that* $Q_n \overset{n\to\infty}{\longrightarrow} Q$ *in ucp. Then, we get* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty < +\infty$ *in ucp and hence the convergences of the long-term swap rate and the long-term yield hold also in ucp. Due to Proposition 4.1 the long-term swap rate is*

$$R_t = \frac{\int_{T_0}^{\infty}\phi(s)\, M(t,s)\; ds}{\delta\, \sum_{i=1}^{\infty}\int_{T_i}^{\infty}\phi(s)\, M(t,s)\; ds}\, , t \geq 0\, . \tag{6.7}$$

Now, we again want to know the long-term yield in this case. It holds

$$\ell_{\cdot} = -\lim_{T\to\infty} T^{-1}\log\biggl(\int_{T}^{\infty}\phi(s)\, M(\,\cdot\,,s)\; ds\biggr) \quad \textit{in ucp.}$$

From Theorem 5.5 we know that $\ell_t \geq 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$ *since* $R_t > 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$ *due to* (6.7) *and the long-term bond price vanishes. Further,* $L_t \geq 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$ *by Theorem 5.5.*

6.2 Long-Term Rates in the Linear-Rational Methodology

The class of linear-rational term structure models is introduced in [15] for the first time. This class presents several advantages: it is highly tractable and offers a very good fit to interest rate swaps and swaptions data since 1997. Further, non-negative interest rates are guaranteed, unspanned factors affecting volatility and risk premia are accommodated, and analytical solutions to swaptions are admitted.

We assume the existence of a state price density, i.e. of a positive adapted process $A := (A_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the price $\Pi(t,T)$ at time t of any time T cashflow C_T is given by

$$\Pi(t,T) = \frac{\mathbb{E}^{\mathbb{P}}[\, A_T C_T |\, \mathcal{F}_t]}{A_t}\, ,\ 0 \leq t \leq T\, , \tag{6.8}$$

for all $T \geq 0$. In particular we suppose that the state price density A is driven by a multivariate factor process $X := (X_t)_{t\geq 0}$ with state space $E \subseteq \mathbb{R}^d$, $d \geq 1$, where

$$dX_t = k\,(\theta - X_t)\, dt + dM_t\, ,\ t \geq 0\, , \tag{6.9}$$

for some $k \in \mathbb{R}_+$, $\theta \in \mathbb{R}^d$, and some martingale $M := (M_t)_{t\geq 0}$ on E. We assume to work with the càdlàg version of X. Next, A is defined as

$$A_t := \exp(-\alpha t)\left(\phi + \psi^\top X_t\right)\,,\ t \geq 0\,, \tag{6.10}$$

with $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^d$ such that $\phi + \psi^\top x > 0$ for all $x \in E$, and $\alpha \in \mathbb{R}$. It holds $\alpha = \sup_{x\in E} \frac{k\,\psi^\top(\theta - X_t)}{\phi+\psi^\top x}$ to guarantee non-negative short rates (cf. equation (6) of [15]). Then, equations (6.8), (6.9), (6.10), together with the fact that $P(T,T) = 1$ for all $T \geq 0$, lead to

$$P(t,T) = \frac{\left(\phi + \psi^\top\theta\right)\exp(-\alpha\,(T-t)) + \psi^\top(X_t - \theta)\exp(-\,(\alpha+k)\,(T-t))}{\phi + \psi^\top X_t} \tag{6.11}$$

for all $0 \leq t \leq T$. Hence, $P_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ and we know by the ratio test that for all $t \geq 0$

$$\alpha_\infty(t) := \sum_{i=1}^\infty \exp(-\alpha\,(T_i - t)) < +\infty\,, \quad \beta_\infty(t) := \sum_{i=1}^\infty \exp(-\,(\alpha+k)\,(T_i-t)) < +\infty\,.$$

Then for all $t \geq 0$ $\mathbb{P}$-a.s.

$$S_\infty(t) \overset{(6.11)}{=} \frac{\left(\phi + \psi^\top\theta\right)\alpha_\infty(t) + \psi^\top(X_t - \theta)\,\beta_\infty(t)}{\phi + \psi^\top X_t} < +\infty\,. \tag{6.12}$$

It follows by Proposition 4.1 that for all $t \geq 0$ $\mathbb{P}$-a.s.

$$R_t \underset{(6.12)}{\overset{(6.11)}{=}} \frac{\left(\phi + \psi^\top\theta\right)\exp(-\alpha\,(T_0 - t)) + \psi^\top(X_t - \theta)\exp(-\,(\alpha+k)\,(T_0-t))}{\delta\left(\left(\phi + \psi^\top\theta\right)\alpha_\infty(t) + \psi^\top(X_t - \theta)\,\beta_\infty(t)\right)}\,.$$

Finally, we want to know the form of the long-term yield in the linear-rational term structure methodology. We define $y := \phi + \psi^\top\theta$ and see that for all $t \geq 0$ holds

$$\log\left[y + \psi^\top\left(\sup_{0\leq s\leq t} X_s - \theta\right)e^{-k(T-t)}\right] \geq \log\left[y + \psi^\top\,(X_t - \theta)\,e^{-k(T-t)}\right]$$

as well as

$$\log\left[y + \psi^\top\,(X_t - \theta)\,e^{-k(T-t)}\right] \geq \log\left[y + \psi^\top\left(\inf_{0\leq s\leq t} X_s - \theta\right)e^{-k\,T}\right].$$

This yields $\mathbb{P}$-a.s. for all $t \geq 0$

$$\begin{aligned}
\sup_{0\leq s\leq t}\left|\alpha + \frac{\log P(s,T)}{T}\right| &\overset{(6.11)}{=} \sup_{0\leq s\leq t}\left|\alpha\frac{s}{T} + \frac{1}{T}\log\left[y + \psi^\top\,(X_s - \theta)\,e^{-k(T-s)}\right]\right| \\
&\leq \sup_{0\leq s\leq t}\left|\alpha\frac{s}{T}\right| + \sup_{0\leq s\leq t}\frac{1}{T}\left|\log\left[y + \psi^\top\,(X_s - \theta)\,e^{-k(T-s)}\right]\right| \\
&\overset{T\to\infty}{\longrightarrow} 0
\end{aligned}$$

because $\sup_{0\leq s\leq t} X_s < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ since X is càdlàg. Hence, we have for all $t \geq 0$ that $\lim_{T\to\infty} \sup_{0\leq s\leq t} Y(s,T) = \alpha$ $\mathbb{P}$-a.s., consequently in probability, i.e. we get $\ell_t = \alpha$ $\mathbb{P}$-a.s. for all $t \geq 0$. In case of α positive, the long-term simple rate explodes due to Theorem 5.1.

Note that the Flesaker-Hughston model can be transferred to the linear-rational term structure methodology and vice versa. In Example 1 of Section 6.1 we can choose $f(t) := \exp(-\alpha t)\,\phi$, $g(t) := \exp(-\alpha t)\,\psi$, and $M_t := X_t$ to get the linear-rational term structure model from the Flesaker-Hughston model. Further, in the second example of Section 6.1, we can set $\phi(s) := \exp(-\alpha s)$ and $M(t,s) := \mathbb{E}^{\mathbb{P}}[\,r(X_s)|\,\mathcal{F}_t]\ , 0 \leq t \leq s$, with $r(x) := \alpha\phi - k\psi^\top\theta + (\alpha+k)\,\psi^\top x\,, x \in E$, to obtain the linear-rational term structure model by the Flesaker-Hughston approach (cf. Section 2.4 of [15]).

A Infinite Sum of Bond Prices

For the study of the long-term swap rate in Section 3 as well as of the relations among the different long-term interest rates in Section 5 we need to obtain some results on the infinite sum S_∞ of bond prices defined in (3.3). The next two statements give insight about the relation between the long-term zero-coupon bond prices and the asymptotic behaviour of the sum of these prices, whereas Lemma A.3 tells us that the long-term simple rate vanishes if the long-term bond price explodes.

Proposition A.1. *If $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, then $P_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. From $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp it follows that $S_\infty(t) < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. We get for all $\epsilon > 0$ and $t \geq 0$ with $C^{\epsilon,t,n} := \{\omega \in \Omega : \sup_{0\leq s\leq t} |P(s,T_n)| > \epsilon\}$

$$\begin{aligned}
\mathbb{P}\big(C^{\epsilon,t,n}\big) &= \mathbb{P}\Big(\sup_{0\leq s\leq t} |S_n(s) - S_{n-1}(s)| > \epsilon\Big) \\
&= \mathbb{P}\Big(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s) + S_\infty(s) - S_{n-1}(s)| > \epsilon\Big) \\
&\leq \mathbb{P}\Big(\sup_{0\leq s\leq t} \big(|S_n(s) - S_\infty(s)| + |S_{n-1}(s) - S_\infty(s)|\big) > \epsilon\Big) \\
&\leq \mathbb{P}\Big(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| + \sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \epsilon\Big) \\
&\leq \mathbb{P}\Big(\Big\{\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| > \frac{\epsilon}{2}\Big\} \cup \Big\{\sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \frac{\epsilon}{2}\Big\}\Big)
\end{aligned}$$

$$\overset{(*)}{\leq} \mathbb{P}\left(\sup_{0\leq s\leq t}|S_n(s)-S_\infty(s)|>\frac{\epsilon}{2}\right)+\mathbb{P}\left(\sup_{0\leq s\leq t}|S_{n-1}(s)-S_\infty(s)|>\frac{\epsilon}{2}\right)$$
$$\overset{n\to\infty}{\longrightarrow} 0$$

due to the ucp convergence of S_n. At $(*)$ we used Satz 1.11 (d) of [19]. Hence $P_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. ☐

Corollary A.2. *If $\mathbb{P}(P_t > 0) > 0$ for some $t \geq 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. This is a direct consequence of Proposition A.1. ☐

Lemma A.3. *If $P = +\infty$, it follows $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. It follows $L(\,\cdot\,,T_n) \overset{n\to\infty}{\longrightarrow} 0$ in ucp by (2.2) and the definition of convergence to $+\infty$ in ucp (cf. Definition B.3). ☐

B UCP Convergence

The definition of uniform convergence on compacts in probability (ucp convergence) can be found in Chapter II, Section 4 of [29]. We repeat this here for the reader's convenience. As before we consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_\infty \subseteq \mathcal{F}$ satisfying the usual hypothesis. All processes are adapted to $\mathbb{F}$.

Definition B.1. *A sequence of processes $(X^n)_{n\in\mathbb{N}}$ converges to a process X uniformly on compacts in probability if, for each $t > 0$, $\sup_{0\leq s\leq t}|X^n_s - X_s|$ converges to 0 in probability, i.e. for all $\epsilon > 0$ it holds*

$$\mathbb{P}\left(\sup_{0\leq s\leq t}|X^n_s - X_s| > \epsilon\right) \overset{n\to\infty}{\longrightarrow} 0\,. \tag{B.1}$$

We write $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp.

Theorem B.2. *Let $(X^n)_{n\in\mathbb{N}}$ and $(Y^n)_{n\in\mathbb{N}}$ be sequences of processes . If $(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} (X, Y)$ in ucp with $\sup_{0\leq s\leq t}|X_s| < +\infty$ and $\sup_{0\leq s\leq t}|Y_s| < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $f(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} f(X, Y)$ in ucp for all $f : \mathbb{R}^2 \to \mathbb{R}$ continuous.*

Proof. Let us define $\nu^n_s := (X^n_s, Y^n_s)$, $\nu_s := (X_s, Y_s)$, and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^2$. We have to show that for all $t \geq 0$ and $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t}|f(\nu^n_s) - f(\nu_s)| > \epsilon\right) \overset{n\to\infty}{\longrightarrow} 0. \tag{B.2}$$

Let $k \geq 0$. Then for all $t \geq 0$ it holds

$$\left\{ \sup_{0 \leq s \leq t} |f(\nu_s^n) - f(\nu_s)| > \epsilon \right\} \subseteq \left\{ \sup_{0 \leq s \leq t} |f(\nu_s^n) - f(\nu_s)| > \epsilon,\ \sup_{0 \leq s \leq t} \|\nu_s\| \leq k \right\} \cup \left\{ \sup_{0 \leq s \leq t} \|\nu_s\| > k \right\}. \tag{B.3}$$

By the Heine-Cantor theorem (cf. Theorem A.1.1 of [8]) it follows from f continuous that f is uniformly continuous on any bounded interval and therefore there exists for the given $\epsilon > 0$ a $\delta > 0$ such that

$$\left\{ \sup_{0 \leq s \leq t} |f(\nu_s^n) - f(\nu_s)| > \epsilon,\ \sup_{0 \leq s \leq t} \|\nu_s\| \leq k \right\} \subseteq \left\{ \sup_{0 \leq s \leq t} \|\nu_s^n - \nu_s\| > \delta,\ \sup_{0 \leq s \leq t} \|\nu_s\| \leq k \right\} \subseteq \left\{ \sup_{0 \leq s \leq t} \|\nu_s^n - \nu_s\| > \delta \right\} \tag{B.4}$$

Substituting (B.4) into (B.3) gives us

$$\left\{ \sup_{0 \leq s \leq t} |f(\nu_s^n) - f(\nu_s)| > \epsilon \right\} \subseteq \left\{ \sup_{0 \leq s \leq t} \|\nu_s^n - \nu_s\| > \delta \right\} \cup \left\{ \sup_{0 \leq s \leq t} \|\nu_s\| > k \right\}.$$

Hence

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} |f(\nu_s^n) - f(\nu_s)| > \epsilon \right) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t} \|\nu_s^n - \nu_s\| > \delta \right) + \mathbb{P}\left(\sup_{0 \leq s \leq t} \|\nu_s\| > k \right). \tag{B.5}$$

Since $\sup_{0 \leq s \leq t} |X_s| < +\infty$ and $\sup_{0 \leq s \leq t} |Y_s| < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, it holds for all $t \geq 0$ that $\mathbb{P}\big(\sup_{0 \leq s \leq t} \|\nu_s\| > k\big) \overset{k \to \infty}{\longrightarrow} 0$. Let first $k \to \infty$ and then $n \to \infty$, to obtain (B.2) from (B.5). $\square$

In order to treat the case of exploding long-term interest rates, we now provide a definition of convergence to $\pm\infty$ in ucp.

Definition B.3. *A sequence of processes $(X^n)_{n \in \mathbb{N}}$ converges to $+\infty$ uniformly on compacts in probability if, for each $t > 0$ and $M > 0$ it holds*

$$\mathbb{P}\left(\inf_{0 \leq s \leq t} X_s^n > M \right) \overset{n \to \infty}{\longrightarrow} 1\,. \tag{B.6}$$

We write $X^n \overset{n \to \infty}{\longrightarrow} +\infty$ in ucp.

Accordingly the sequence of processes $(X^n)_{n \in \mathbb{N}}$ converges to $-\infty$ uniformly on compacts in probability if, for each $t > 0$ and $M > 0$ it holds

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} X_s^n < -M \right) \overset{n \to \infty}{\longrightarrow} 1\,. \tag{B.7}$$

Then, we write $X^n \overset{n \to \infty}{\longrightarrow} -\infty$ in ucp.

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