AFFINE HJM FRAMEWORK ON S_d^+ AND LONG-TERM YIELD

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Abstract. We develop the HJM framework for forward rates driven by affine processes on the state space of symmetric positive matrices. In this setting we find a representation for the long-term yield and investigate the yield's asymptotic behaviour. This generalises the results of [38] and [6], where the long-term yield is investigated under no-arbitrage assumptions in a HJM setting using Brownian motions and Lévy processes respectively.

1. Introduction

Term structure modelling is a classical problem in mathematical finance. The relevance of the topic may be ascribed to the overall size of the market for interest rate related products. According to the Bank for International Settlements, [3], the outstanding notional amount of OTC derivatives accross different asset classes as of December 2012 was estimated to be about 632 trillion dollars. Interest rate products are playing a major role in this figure with an estimated outstanding notional amount of about 489 trillion dollars. Given these figures, it is not surprising that interest rate modelling has always been one of the most relevant problems both from a practitioner and academic point of view.

In this paper, we provide an extension of the classical Heath-Jarrow-Morton framework to a setting where the market is driven by semimartingale taking values on the cone S_d^+ of positive semidefinite symmetric $d \times d$ matrices. This class of stochastic processes has appealing features and is increasingly studied in finance research, in particular for modelling multivariate stochastic volatilities in equity and fixed income models, cf. e.g. [5], [16], [17], [18], [30], [46], and [49]. In particular, it allows to model a whole family of factors which share non-linear links among each other, providing a more realistic description of the market. In many situations, the presence of stochastic correlations among factors does not come at the cost of a loss of analytical tractability, as these processes are affine, in the sense of [13]. The class of affine processes on S_d^+, i.e. stochastically continuous Markov processes with the feature that the Laplace transform can be represented as an exponential-affine function, was introduced to applications in finance by [29] and [30] in the form of Wishart processes, a particular affine process first described by Bru in [9]. Theoretical background to affine processes on S_d^+ can be found, among other publications, in [12], [13], [14], [20], [27], [31], [44] and [43]. A first application of Wishart processes for short rates modelling is given in [26], while a Libor model using affine processes is constructed in [15]. Here we consider for the first time an affine HJM framework on S_d^+, where we develop formulas for forward rates, short rates, and continuously compounded spot rates as well as determine the HJM condition on the drift. Note also that we allow general affine processes on S_d^+, i.e. we admit jumps. This setting provides a flexible and synthetic way of taking in

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account the influence of a large number of factor on interest rates dynamics and represents a further contribution in capturing the dependence structure affecting the interest rates evolution.

Equipped with the affine HJM setting on S_d^+, we devote the second part of the present article to the study of the long-term yield. Long-term interest rates are particularly relevant for the pricing and hedging of long-term fixed-income securities, pension funds, life and accident insurances, or interest rate swaps with a very long time to maturity. Thus, the modelling of long-term interest rates is the topic of several contributions which however do not provide a unique definition of long-term interest rates or yield. In Section 4 we provide a brief discussion on the different conventions concerning the time to maturity defining the concept of long-term yield that we found in the literature. Several studies address the topic from a more mathematical or a more macroeconomic point of view. The macroeconomic approach [41] examines the impact of monetary and fiscal policies on long-term interest rates and rejects the hypothesis that long-term interest are overly sensitive to short-term rates. The article [32] also studies the impact of macroeconomic news and monetary policy surprises on long-term yields and presents evidence that these factors have significant effects on short-term as well as on long-term interest rates. The work [34] describes a joint model of macroeconomic and yield curve dynamics where the continuously compounded spot rate is an affine function dependent on macroeconomic state variables. With the help of this model the influence of macroeconomic effects on the long-term yield can be measured. The finding of a model that jointly characterises the behaviour of the yield curve and macroeconomic variables as well as state results for the short-term and long-term interest rates is also the subject of [1] and [19]. In [1] a vector autoregression model is used to describe the relationship between interest rates and macroeconomy, whereas [19] uses a latent factor model with the inclusion of macroeconomic variables to model the yield curve. Another macroeconomic approach is presented in [11] where several economic factors are studied with their respective influence on asset pricing. One of these factors is the long-term yield in terms of long-term government bonds. In [39] the yield curve is modeled by a three-factor model, where the interest rates can be described with the help of three underlying latent factors which are employed in order to explain the empirical result of falling long-term yields.

Mathematical approaches consider the long-term yield as an interest rate with time to maturity tending to infinity. In the textbook [10] as well as in [6], [38], and [53], the long-term yield is defined as the limit of the continuously compounded spot rate. In this paper we adopt this definition. The respective form of the long-term yield then depends on the chosen interest rate model, whereas there can be made some universal statements concerning the asymptotic behaviour of yields in an arbitrage-free market, independent of the chosen model.

One of the most important results concerning the asymptotic behaviour of yields is that in an arbitrage-free market, long-term zero-coupon rates can never fall, as first stated in [22], consequently referred to as *DIR-Theorem*. This result was made rigorous by [45]. An alternative proof using a different definition of arbitrage can be found in [50]. Then, [35] provided a generalisation of the proof of the *DIR-Theorem*, where the assumption of the existence of an equivalent martingale measure is used instead of an arbitrage strategy and hence some measurability conditions can be omitted. The assumption of the existence of an equivalent martingale measure is relaxed in [37]. Finally, [28] generalised the theorem in the sense that it is shown that the limit superior of zero-coupon rates and forward rates never fall, so the existence of the respective long-term limits is not required.

Concrete computations of the long-term yield as limit of the standard yield have been done in [10], [38], [52], [53] in a Brownian motion setting and more recently in [6] in a general Lévy setting. In [6] an explicit form for the long-term yield is provided that takes in account also the impact of jumps on the long-term behaviour. Since it is very important to provide explicit models for the long-term yield for several applications, we study here the long-term yield in an HJM framework driven by a general affine process on S_d^+. This setting presents the main advantage that the forward curve can be described by taking in account a richer interdependence structure among factors, which cannot be caught by other drivers, e.g. Lévy processes. This provides a flexible way of describing the impact of different risk factors and of their correlations on the long-term yield. Under some integrability and measurability conditions on the parameters, we are able to obtain an explicit form of the long-term yield, which results to be independent of the underlying probability measure. This extends a result of Section 2.2 in [38] in a Brownian motion setting. Moreover, we prove that in our context jumps in the dynamics of the yield do not impact the long-term behaviour.

The paper is structured as follows. In the next section we present the main properties of affine processes on S_d^+ as well as features that are important in the course of this paper. Then, in Section 3 analytical expressions for different interest rates are developed under the HJM framework with an affine process X on S_d^+ as stochastic driver of the forward rate. In Section 4 we provide a representation of the long-term yield in the HJM framework on S_d^+.

2. Affine Processes on S_d^+

Affine processes were initially studied by [21] and later fully characterised by [20] on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ with $m, n \in \mathbb{N}$. The theoretical framework for affine processes on the state space S_d^+, can be found in extensive forms in [13] and [42]. In this section, we state, for the reader‘s convenience, the results of these works which are used in the course of this paper as well as the basic required notations. In general, for the stochastic background and notation we refer to [48]. We write $\mathbb{R}_+ = [0, \infty[$, $\mathbb{R}_{++} =]0, \infty[$, $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, $\mathbb{R} =]-\infty, +\infty[$, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Let $d \in \mathbb{N}$. Then, $\mathcal{M}_d$ denotes the set of all $d \times d$ matrices with entries in $\mathbb{R}$, S_d is the space of symmetric $d \times d$ matrices with entries in $\mathbb{R}$, S_d^- encompasses all symmetric $d \times d$ negative semidefinite matrices with entries in $\mathbb{R}$, and S_d^+ stands for the cone of symmetric $d \times d$ positive semidefinite matrices with entries in $\mathbb{R}$ which induces a partial order relation on S_d:

$$\text{For } x, y \in S_d \text{ it is } x \preceq y \text{ if } y - x \in S_d^+ \,.$$

The space $\mathcal{M}_d$ is endowed with the scalar product $A \cdot B := \operatorname{Tr}\left[A^\top B\right]$ for $A, B \in \mathcal{M}_d$, where $\operatorname{Tr}[AB]$ denotes the trace of the matrix AB.

Throughout this paper, given $A \subseteq \mathcal{M}_d$, $\mathcal{B}(A)$ denotes the Borel σ-algebra on A and $b(A)$ the Banach space of bounded real-valued Borel-measurable functions f on A with norm $\|f\|_\infty = \sup_{x \in A} |f(x)|$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ be a filtered probability space with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions of completeness and right-continuity and $X := (X_t)_{t \geq 0}$ a stochastic process on this probability space. For $x \in S_d^+$, $\mathbb{P}_x$ is a probability measure such that $\mathbb{P}_x(X_0 = x) = 1$. Given $t > 0$, $X_{t-} := \lim_{s \uparrow t} X_s$, we define

$$\Delta X_t := X_t - X_{t-} \,, \tag{2.1}$$

the jump at t, $\Delta X_0 \equiv 0$.

Next, we define the transition probabilities for all $t \geq 0$ as:

$$p_t : S_d^+ \times \mathcal{B}\big(S_d^+\big) \to [0, 1] \,, (x, B) \mapsto \mathbb{P}_x(X_t \in B) \,.$$

Further, let $(P_t)_{t\geq 0}$ be a semigroup such that

$$P_t f(x) := \int_{S_d^+} f(\xi)\; p_t(x, d\xi) = \mathbb{E}_x[f(X_t)]\,, x \in S_d^+\,, \tag{2.2}$$

where $f \in b\big(S_d^+\big)$.

We consider a time-homogeneous Markov process X with state space S_d^+, i.e. the Markov property holds for all $A \in \mathcal{B}\big(S_d^+\big)\,, x \in S_d^+$, and $s, t \geq 0$ (cf. Definition 17.3 in [40]):

$$\mathbb{P}_x(X_{t+s} \in A \mid \mathcal{F}_s) = p_t(X_s, A) \quad \mathbb{P}_x \text{ - a.s.}$$

Next, we want to define the characteristics of an affine process on S_d^+ (cf. Definition 2.1 in [13]).

Definition 2.1. *A Markov process X with values in S_d^+ is called* affine *if the following two properties hold:*

(i) *It is stochastically continuous, i.e. it holds for all $t \geq 0$ and all $\epsilon > 0$:*

$$\lim_{s\to t} \mathbb{P}_x(\|X_s - X_t\| > \epsilon) = 0\,.$$

(ii) *Its Laplace transform has exponential-affine dependence on the initial state, i.e. the following equation holds for all $t \geq 0$ and $u, x \in S_d^+$:*

$$P_t e^{-\operatorname{Tr}[ux]} \overset{(2.2)}{=} \int_{S_d^+} e^{-\operatorname{Tr}[u\xi]}\, p_t(x, d\xi) = e^{-\phi(t,u) - \operatorname{Tr}[\psi(t,u)x]}\,, \tag{2.3}$$

for some functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$.

From the stochastic continuity of X follows directly the weak convergence of the distributions $p_t(x, \cdot)$, $t \geq 0$, i.e. it holds for all $t \geq 0$ (cf. Satz 5.1 in [4]):

$$\lim_{s\to t} p_s(x, \cdot) = p_t(x, \cdot)\ .$$

Note, that due to the non-negativity of X the Laplace transform is well-defined and can be used to characterise an affine process. Further, in consequence of the stochastic continuity of the process according to Proposition 3.4 in [13], the process X is regular in the sense of Definition 2.2 in [13].

As well we consider that the affine Markov process is conservative, that means that the process will remain almost surely on the state space S_d^+ all the time.

Definition 2.2. *The affine process X is called* conservative *if for all $t \geq 0$ the following condition holds*

$$p_t\big(x, S_d^+\big) = 1\,,$$

i.e. $X_t \in S_d^+$ $\mathbb{P}_x$-a.s.

Now, we are able to introduce the so-called admissible parameter set which generalises the concept of Lévy triplet to the setting of affine processes on S_d^+ (cf. Definition 3.1 in [42]).

Definition 2.3. *An* admissible parameter set (α, b, B, m, μ) *consists of*

(i) *a linear diffusion coefficient $\alpha \in S_d^+$,*
(ii) *a constant drift term $b \in S_d^+$ which satisfies*

$$b \succeq (d-1)\,\alpha\,,$$

(iii) *a Borel measure m on $S_d^+\backslash\{0\}$ to represent the constant jump term*

$$\int_{S_d^+\backslash\{0\}} (\|\xi\| \wedge 1)\; m(d\xi) < \infty\,, \tag{2.4}$$

(iv) *a linear jump coefficient* $\mu : S_d^+ \setminus \{0\} \to S_d^+ \setminus \{0\}$ *which is a* σ*-finite measure and satisfies*

$$\int_{S_d^+\setminus\{0\}} (\|\xi\| \wedge 1)\ \mu(d\xi) < \infty\,, \tag{2.5}$$

(v) *a linear drift* $B : S_d^+ \to S_d^+$ *that satisfies the condition*

$$\operatorname{Tr}[B(x)\,u] \geq 0 \text{ for all } x, u \in S_d^+ \text{ with } \operatorname{Tr}[xu] = 0\,.$$

Theorem 2.1. *Suppose* X *is a conservative affine process on* S_d^+ *with* $d \geq 2$*. Then* X *is regular and has the Feller property. Moreover, there exists an admissible parameter set* (α, b, B, m, μ) *such that* $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ *and* $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$ *in* (2.3) *solve the generalised Riccati differential equations for* $u \in S_d^+$

$$\partial_t \phi(t,u) = F(\psi(t,u))\,, \quad \phi(0,u) = 0\,, \tag{2.6}$$

$$\partial_t \psi(t,u) = R(\psi(t,u))\,, \quad \psi(0,u) = 0\,, \tag{2.7}$$

with

$$F(u) := \operatorname{Tr}[bu] - \int_{S_d^+\setminus\{0\}} \left(e^{-\operatorname{Tr}[u\xi]} - 1\right) m(d\xi)\ , \tag{2.8}$$

$$R(u) := -2\,u\,\alpha\,u + B^\top(u) - \int_{S_d^+\setminus\{0\}} \left(e^{-\operatorname{Tr}[u\xi]} - 1\right) \mu(d\xi)\ . \tag{2.9}$$

Conversely, let (α, b, B, m, μ) *be an admissible parameter set and* $d \geq 2$*. Then there exists a unique conservative affine process* X *on* S_d^+ *such that the affine property* (2.3) *holds for all* $t \geq 0$ *and* $u, x \in S_d^+$ *with* $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ *and* $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$ *given by* (2.6) *and* (2.7)*.*

Proof. Cf. Theorem 2.4 of [13] and Theorem 4.1 of [42]. □

Besides the admissible parameter set, we need to define the matrix variate Brownian motion for the representation of the affine process X (cf. Definition 3.23 in [47]).

Definition 2.4. *A* matrix variate Brownian motion $W \in \mathcal{M}_d$ *is a matrix consisting of* d^2 *independent, one-dimensional Brownian motions* $W_{ij}, 1 \leq i, j \leq d$*.*

Remark 1. *By* (3.3) *of* [42] *we obtain that in the case of* $d \geq 2$*, the affine process* X *has only jumps of finite variation, i.e for all* $t \geq 0$

$$\int_0^t \int_{S_d^+\setminus\{0\}} \|\xi\|\ \mu^X(ds, d\xi) < \infty\,. \tag{2.10}$$

Now, we can state the following representation of X.

Theorem 2.2. *Let* X *be a conservative affine process on* S_d^+*,* $d \geq 2$*, with admissible parameter set* (α, b, B, m, μ)*, where* $Q \in \mathcal{M}_d$ *such that* $Q^\top Q = \alpha$*. Then there exists a matrix Brownian motion* $W \in \mathcal{M}_d$ *such that* X *admits the following representation:*

$$X_t = x + \int_0^t (b + B(X_s))\,ds + \int_0^t \left(\sqrt{X_s} dW_s Q + Q^\top dW_s^\top \sqrt{X_s}\right) + \int_0^t \int_{S_d^+\setminus\{0\}} \xi\, \mu^X(ds, d\xi)\,, \tag{2.11}$$

where $\mu^X(ds, d\xi)$ *is the random measure associated with the jumps of* X*, having the compensator*

$$\nu(dt, d\xi) := (m(d\xi) + \operatorname{Tr}[X_t\, \mu(d\xi)])\, dt\,. \tag{2.12}$$

Proof. Cf. Theorem 3.4 in [42]. □

Note, that it is possible to choose Q this way since $Q^\top Q \in S_d^+$ for all $Q \in \mathcal{M}_d$ due to Theorem 2.2 (ix) in [47].

Remark 2. *If in* Theorem 2.2 *we have $b = \delta\alpha$ with $\delta \geq 0$, $B(z) = Mz + zM^\top$ with $M \in \mathcal{M}_d$, and there are no jumps, the process X is a Wishart process, cf.* [9].

Throughout this paper we consider X to be a conservative, regular, affine process on the state space S_d^+ with $d \geq 2$, hence X can be represented by equation (2.11). Furthermore, the linear drift coefficient B is of the form

$$B(z) = Mz + zM^\top + \Gamma(z)\,,\; z \in S_d^+\,, \tag{2.13}$$

where $M \in \mathcal{M}_d$ and $\Gamma : S_d \to S_d$ is linear satisfying $\Gamma\big(S_d^+\big) \subseteq S_d^+$ to encompass a wider range of affine processes (cf. (2.30) in [13]).

Note, that in the case of X being not conservative, all subsequent calculations and the consequential results are still valid, as long as another set of admissible parameters is used with an additional constant killing rate term $c \in \mathbb{R}_+$ and an additional linear killing rate coefficient $\gamma \in S_d^+$. In the case of $d = 1$, the parameter set has to be extended by a truncation function for compensating the infinite variation part of the jumps. The most general admissible parameter set, encompassing the case of X being not conservative on a state space with dimension $d = 1$, is stated in Definition 2.3 in [13].

3. Affine HJM Framework on S_d^+

We now provide a HJM framework to model the forward curve using affine processes on S_d^+ in the setting outlined in Section 2.

By a T-maturity zero-coupon bond we mean a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t \leq T$ is denoted by $P(t,T)$ and the bond market satisfies the following hypotheses: (1) there exists a frictionless market for T-bonds for every maturity $T \geq 0$; (2) $P(T,T) = 1$ for every $T \geq 0$; (3) for each fixed t, the zero-coupon bond price $P(t,T)$ is differentiable with respect to the maturity T.

The money market account is $\beta_t := \exp\Big(\int_0^t r_s\, ds\Big)$ with r_t denoting the short rate at time t. We set $\Delta^2 := \{(t,T) \in \mathbb{R}_+ \times \mathbb{R}_+,\; t \leq T\}$ and assume the forward rates $f : \Omega \times \Delta^2 \to \mathbb{R}$ to evolve for every maturity $T > 0$ according to

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\; ds + \int_0^t \mathrm{Tr}[\sigma(s,T)\, dX_s]\,,\; 0 \leq t \leq T, \tag{3.1}$$

where X is an affine conservative process with representation (2.11) for a given initial value $x \in S_d^+$. Since we fix the initial value $X_0 = x$, from now on we write $\mathbb{P}$ for $\mathbb{P}_x$. We impose the following conditions on the drift $\alpha : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and the volatilities $\sigma_{ij} : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+,\; i,j \in \{1, \dots, d\}$:[1]

Assumption 1.

- $\alpha := \alpha(\omega, s, u) : (\Omega \times \mathbb{R}_+ \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+)) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *is jointly measurable.*
- *For all $T \geq 0$:*
$$\int_0^T \int_0^T |\alpha(s,u)|\; ds\, du < \infty \;\; \mathbb{P}\text{-a.s.}$$
- $\alpha(s,u) = 0$ *for all $s \geq u$, $u \in \mathbb{R}_+$.*
- *For all $s, u \in \mathbb{R}_+$ and a.e. $\omega \in \Omega$: $\sigma(s,u) \in S_d$, i.e. $\sigma(s,u)$ is a symmetric $d \times d$ matrix.*

[1]For α and σ we write the shortened version $\alpha(s,T) := \alpha(\omega, s, T)$ and $\sigma(s,T) := \sigma(\omega, s, T)$.

- $\sigma_{ij} := \sigma_{ij}(\omega, s, u) : (\Omega \times \mathbb{R}_+ \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+)) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *are jointly measurable for all* $i, j \in \{1, \dots, d\}$.
- *For all* $T \geq 0$: $(\alpha(s,T))_{s \in [0,T]}$ *and* $(\sigma(s,T))_{s \in [0,T]}$ *are adapted.*
- *For all* $T \geq 0$:
$$\sup_{s,u \leq T} \|\sigma(s,u)\| < \infty \ \ \mathbb{P}\text{-a.s.}$$
- *For all* $i, j \in \{1, \dots, d\} : \sigma_{ij} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is càglàd in both components.*
- *For all* $i, j \in \{1, \dots, d\} : \sigma_{ij}(s,u) > 0$ *for all* $u \in \mathbb{R}_+$, $s \in [0, u[$.
- *For all* $i, j \in \{1, \dots, d\} : \sigma_{ij}(s,u) = 0$ *for all* $s \geq u$, $u \in \mathbb{R}_+$.

Due to Assumption 1 the forward rate process is well-defined in (3.1). Note that other integrability conditions can be chosen to guarantee that the integrals in (3.1) are well-defined. In this case the results of the paper will also apply under technical modifications of the proofs.

Proposition 3.1. *If X is a conservative affine process and* Assumption 1 *holds, then for every maturity $T > 0$ the zero-coupon bond price follows a process of the form:*

$$\begin{aligned} P(t,T) = P(0,T) &+ \int_0^t P(s,T)\,(r_s + A(s,T))\,ds \\ &+ 2\int_0^t P(s,T)\,\mathrm{Tr}\Big[\Sigma(s,T)\sqrt{X_s}\,dW_s Q\Big] \\ &+ \int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}}\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\left(\mu^X - \nu\right)(ds, d\xi)\,, \end{aligned} \tag{3.2}$$

for $t \leq T$, where

$$\Sigma(s,T) := -\int_s^T \sigma(s,u)\,du \tag{3.3}$$

is the T-bond volatility and

$$A(t,T) := -\int_t^T \alpha(t,u)\,du - F(-\Sigma(t,T)) - \mathrm{Tr}[R(-\Sigma(t,T))\,X_t]\,, \tag{3.4}$$

where F and R are given by (2.8), (2.9) *respectively.*

Proof. Let us introduce for every maturity $T > 0$ the quantity

$$Z(t,T) := -\int_t^T f(t,u)\,du\,, \tag{3.5}$$

for all $0 \leq t \leq T$. From the dynamics of the forward rate (3.1) we deduce that for all $T > 0$

$$Z(t,T) \overset{(3.5)}{=} -\int_t^T f(0,u)\,du - \int_t^T\int_0^t \alpha(s,u)\,ds\,du - \int_t^T\int_0^t \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du\,, \tag{3.6}$$

for all $0 \leq t \leq T$. Let us observe that for all $s \geq 0$

$$r_u := f(u,u) \overset{(3.1)}{=} f(0,u) + \int_0^u \alpha(s,u)\,ds + \int_0^u \mathrm{Tr}[\sigma(s,u)\,dX_s]\,. \tag{3.7}$$

By Assumption 1, the Fubini theorem for integrable functions (cf. Theorem 14.16 in Chapter 14 of [40]) and the stochastic Fubini theorem (cf. Theorem 65 in Chapter IV of [48]) we have

$$-\int_t^T\int_0^t \alpha(s,u)\,ds\,du = -\int_0^t\int_s^T \alpha(s,u)\,du\,ds + \int_0^t\int_0^u \alpha(s,u)\,ds\,du\,, \tag{3.8}$$

and similarly

$$-\int_t^T\int_0^t \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du = -\int_0^t\int_s^T \mathrm{Tr}[\sigma(s,u)\,du\,dX_s] + \int_0^t\int_0^u \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du\,. \tag{3.9}$$

Furthermore, note that for all $T>0$

$$Z(0,T) = -\int_0^T f(0,u)\ du = -\int_t^T f(0,u)\ du - \int_0^t f(0,u)\ du\,. \tag{3.10}$$

By combining (2.11), (3.3), (3.6), (3.7), (3.8), (3.9), and (3.10), we derive the following identity

$$\begin{aligned}
Z(t,T) &\overset{(3.6)}{\underset{(3.10)}{=}} Z(0,T)+\int_0^t f(0,u)\ du - \int_t^T\int_0^t \alpha(s,u)\ ds\,du - \int_t^T\int_0^t \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du \\
&\overset{(3.7)}{=} Z(0,T)+\int_0^t r_s\,ds - \int_0^t\int_s^T \alpha(s,u)\ du\,ds - \int_0^t\int_s^T \mathrm{Tr}[\sigma(s,u)\ du\,dX_s]\ . \\
&\overset{(2.11)}{\underset{(3.3)}{=}} Z(0,T)+\int_0^t r_s\,ds - \int_0^t\int_s^T \alpha(s,u)\ du\,ds \\
&\quad + \int_0^t \mathrm{Tr}\Big[\Sigma(s,T)\left(\sqrt{X_s}\,dW_s\,Q + Q^\top dW_s^\top \sqrt{X_s}\right)\Big] \\
&\quad + \int_0^t \mathrm{Tr}[\Sigma(s,T)\,(b+B(X_s))]\,ds + \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\,\mu^X(ds,d\xi)\ .
\end{aligned}$$

Note that in general for $A,B\in\mathcal{M}_d$ with A symmetric, i.e. $A\in S_d$, it holds

$$\begin{aligned}
\mathrm{Tr}\big[A\left(B+B^\top\right)\big] &= \mathrm{Tr}[AB]+\mathrm{Tr}\big[AB^\top\big] = \mathrm{Tr}[AB]+\mathrm{Tr}\Big[(BA)^\top\Big] \\
&= \mathrm{Tr}[AB]+\mathrm{Tr}\,[BA] = 2\,\mathrm{Tr}[AB]\ .
\end{aligned} \tag{3.11}$$

Therefore, we get due to $\sigma(s,t)\in S_d$ for all $s,t\geq 0$ that

$$\begin{aligned}
Z(t,T) &\overset{(3.11)}{=} Z(0,T)+\int_0^t r_s\,ds - \int_0^t\int_s^T \alpha(s,u)\ du\,ds + 2\int_0^t \mathrm{Tr}\Big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s\,Q\Big] \\
&\quad + \int_0^t \mathrm{Tr}[\Sigma(s,T)\,(b+B(X_s))]\,ds + \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\,\mu^X(ds,d\xi)\ .
\end{aligned} \tag{3.12}$$

Note that for all $0\leq t\leq T$

$$\Delta Z(t,T) = \mathrm{Tr}\,[\Sigma(t,T)\,\Delta X_t]\ . \tag{3.13}$$

With the help of (3.12) and the fact that

$$\langle W_{lm}, W_{ru}\rangle_s = \begin{cases} s & \text{if } l=r \text{ and } m=u, \\ 0 & \text{else,} \end{cases} \tag{3.14}$$

we can calculate the quadratic variation of Z for all $T>0$ as follows

$$\begin{aligned}
\langle Z(\,\cdot\,,T)\rangle_t^c &= \left\langle \mathrm{Tr}\left[\int_0^{\cdot} \Sigma(s,T)\,\sqrt{X_s}\,dW_s\,Q\right]\right\rangle_t \\
&= \Bigg\langle 2\sum_{i,k,l,m}\int_0^{\cdot} \Sigma(s,T)_{ik}\,\sqrt{X_{kl,s}}\,dW_{lm,s}\,Q_{mi}\,, \\
&\qquad\qquad 2\sum_{p,q,r,u}\int_0^{\cdot} \Sigma(s,T)_{pq}\,\sqrt{X_{qr,s}}\,dW_{ru,s}\,Q_{up}\Bigg\rangle_t
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.14)}{=} 4 \sum_{i,k,l,m,p,q} \int_0^t \Sigma(s,T)_{ik} \sqrt{X_{kl,s}}\, Q_{mi}\, \Sigma(s,T)_{pq} \sqrt{X_{ql,s}}\, Q_{mp}\, ds \\
&= 4 \sum_{m,i,k,l,q,p} \int_0^t Q_{mi}\, \Sigma(s,T)_{ik} \sqrt{X_{kl,s}} \sqrt{X_{lq,s}}\, \Sigma(s,T)_{qp}\, Q^\top_{pm}\, ds \\
&= 4 \int_0^t \mathrm{Tr}\big[Q\, \Sigma(s,T)\, X_s\, \Sigma(s,T)\, Q^\top\big]\; ds\,. \qquad (3.15)
\end{aligned}
$$

Further, we see that due to equation (2.27) of [13] for all $u \in S_d^+$ and a process Y on S_d^+ it holds

$$
\mathrm{Tr}\big[B^\top(u)\, Y\big] = \mathrm{Tr}[B(Y)\, u]\;, \qquad (3.16)
$$

where B is defined according to (2.13) and therefore for all $0 \le t \le T$ and $\alpha = Q^\top Q$

$$
\begin{aligned}
&\int_0^t \mathrm{Tr}[\Sigma(s,T)\,(b + B(X_s))] + 2\, \mathrm{Tr}\big[Q\, \Sigma(s,T)\, X_s\, \Sigma(s,T)\, Q^\top\big]\, ds \\
&\qquad + \int_0^t \int_{S_d^+ \setminus \{0\}} \Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\; \nu(ds, d\xi) \\
&\underset{(3.16)}{\overset{(2.12)}{=}} -\int_0^t \mathrm{Tr}\big[-\Sigma(s,T)\, b - 2\,\Sigma(s,T)\, \alpha \Sigma(s,T)\, X_s + B^\top(-\Sigma(s,T))\, X_s\big]\, ds \\
&\qquad + \int_0^t \int_{S_d^+ \setminus \{0\}} \Big(e^{-\,\mathrm{Tr}[-\Sigma(s,T)\,\xi]} - 1\Big)\; m(d\xi)\, ds \\
&\qquad + \int_0^t \mathrm{Tr}\left[X_s \int_{S_d^+ \setminus \{0\}} \Big(e^{-\,\mathrm{Tr}[-\Sigma(s,T)\,\xi]} - 1\Big)\; \mu(d\xi)\right] ds \\
&\underset{(2.8)}{\overset{(2.9)}{=}} \int_0^t \big(-F(-\Sigma(s,T)) - \mathrm{Tr}[R(-\Sigma(s,T))\, X_s]\big)\, ds\,. \qquad (3.17)
\end{aligned}
$$

Now, we apply Itô's formula on $P(t,T) := \exp(Z(t,T))$ for every maturity $T > 0$ (cf. Definition 1.4.2 of [8]) and obtain

$$
\begin{aligned}
P(t,T) &= P(0,T) + \int_0^t P(s-,T)\; dZ(s,T) + \frac{1}{2} \int_0^t P(s,T)\; d\,\langle Z(\,\cdot\,,T)\rangle^c_s \\
&\quad + \sum_{0<s\le t}^{\Delta Z(s,T)\neq 0} \Big[e^{Z(s,T)} - e^{Z(s-,T)} - \Delta Z(s,T)\, e^{Z(s-,T)}\Big] \\
&\underset{(3.15)}{\overset{(3.13)}{=}} P(0,T) + \int_0^t P(s-,T)\; dZ(s,T) \\
&\quad + 2 \int_0^t P(s,T)\, \mathrm{Tr}\big[Q\, \Sigma(s,T)\, X_s\, \Sigma(s,T)\, Q^\top\big]\; ds \\
&\quad + \sum_{0\le s\le t}^{\Delta X_s \neq 0} \Big[e^{Z(s,T)} - e^{Z(s-,T)} - \mathrm{Tr}[\Sigma(s,T)\, \Delta X_s]\, e^{Z(s-,T)}\Big] \\
&\underset{(2.1)}{\overset{(3.12)}{=}} P(0,T) + 2 \int_0^t P(s,T)\, \mathrm{Tr}\Big[\Sigma(s,T) \sqrt{X_s}\, dW_s\, Q\Big] \\
&\quad + \int_0^t P(s,T) \left(r_s - \int_s^T \alpha(s,u)\, du\right)\, ds \\
&\quad + \int_0^t P(s,T)\, \mathrm{Tr}[\Sigma(s,T)\,(b + B(X_s))]\; ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\;\mu^X(ds,d\xi)\\
&+2\int_0^t P(s,T)\,\mathrm{Tr}\big[Q\,\Sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\big]\;ds\\
&+\sum_{0\le s\le t}^{\Delta X_s\neq 0}\Big[e^{\Delta Z(s,T)}P(s-,T)-P(s-,T)-\mathrm{Tr}[\Sigma(s,T)\,\Delta X_s]\,P(s-,T)\Big]\\
\overset{(3.13)}{=}\;& P(0,T)+2\int_0^t P(s,T)\,\mathrm{Tr}\Big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s\,Q\Big]\\
&+\int_0^t P(s,T)\left(r_s-\int_s^T\alpha(s,u)\,du\right)\,ds\\
&+\int_0^t P(s,T)\,\mathrm{Tr}[\Sigma(s,T)\,(b+B(X_s))]\;ds\\
&+\int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\;\mu^X(ds,d\xi)\\
&+2\int_0^t P(s,T)\,\mathrm{Tr}\big[Q\,\Sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\big]\;ds\\
&+\int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}}\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1-\mathrm{Tr}[\Sigma(s,T)\,\xi]\Big)\,\mu^X(ds,d\xi)\\
\overset{(3.17)}{=}\;& P(0,T)+\int_0^t P(s-,T)\,(r_s+A(s,T))\;ds\\
&+2\int_0^t P(s,T)\,\mathrm{Tr}\Big[\Sigma(s,T)\,\sqrt{X_s}\,dW_sQ\Big]\\
&+\int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}}\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\Big)\,\big(\mu^X-\nu\big)\,(ds,d\xi)\;.
\end{aligned}
$$

Note that we are able to combine the measures $\mu^X(ds,d\xi)$ and $\nu(ds,d\xi)$ to $\big(\mu^X-\nu\big)(ds,d\xi)$ because of Proposition 1.28 of Chapter II in [36], since the affine process X has only jumps of finite variation (cf. (2.10)) and Assumption 1 guarantees that all integrals above are finite. □

Remark 3. *The bond-price process $P(t,T)\,,0\le t\le T$, can be rewritten the following way:*

$$
\begin{aligned}
P(t,T)=P(0,T)&+\int_0^t P(s,T)\,(r_s+C(s,T))\;ds+\int_0^t P(s-,T)\,\mathrm{Tr}[\Sigma(s,T)\,dX_s]\\
&+\int_0^t\int_{S_d^+\setminus\{0\}}P(s-,T)\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1-\mathrm{Tr}[\Sigma(s,T)\,\xi]\Big)\big(\mu^X-\nu\big)(ds,d\xi)\,,
\end{aligned}
\tag{3.18}
$$

with for all $0\le t\le T$

$$
C(t,T):=A(t,T)-\mathrm{Tr}[\Sigma(t,T)\,(b+B(X_t))]-\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\Sigma(t,T)\,\xi]\,(m(d\xi)+\mathrm{Tr}[X_t\mu(d\xi)])\,,
\tag{3.19}
$$

where $A(t,T)$ is defined in (3.4).

Proof. Using the representation of the bond-price process (3.2), the representation of the conservative affine process (2.11), and (3.19) it follows (3.18). We again use Proposition 1.28 of Chapter II in [36] to combine the measures $\mu^X(ds, d\xi)$ and $\nu(ds, d\xi)$. $\square$

As an immediate consequence of representation (3.2) for the bond price, we obtain the following corollary.

Corollary 3.1. *For every maturity $T > 0$, the discounted zero-coupon bond price follows a process of the formula*

$$\begin{aligned}\frac{P(t,T)}{\beta_t} &= P(0,T) + \int_0^t \frac{P(s,T)}{\beta_s} A(s,T)\, ds + 2\int_0^t \frac{P(s,T)}{\beta_s} \operatorname{Tr}\Big[\Sigma(s,T)\sqrt{X_s}\, dW_s Q\Big] \\ &\quad + \int_0^t \frac{P(s-,T)}{\beta_s} \int_{S_d^+\setminus\{0\}} \Big(e^{\operatorname{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\left(\mu^X - \nu\right)(ds, d\xi)\,, \quad (3.20)\end{aligned}$$

for all $t \le T$.

Proof. This follows directly from the definition of the money market account and Proposition 3.1. $\square$

We now investigate the restrictions on the dynamics (3.1) under the assumption of no arbitrage. Let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure. By Theorem 3.12 of [7] there exists $\gamma \in \mathcal{M}_d$ with $\int_0^t \|\gamma_s\|^2\, ds < \infty$ for all $t \ge 0$ such that $W_t^* = W_t - \int_0^t \gamma_s\, ds$, $t \ge 0$, is a matrix variate Brownian motion under $\mathbb{Q}$ and an $\mathcal{F}_t \otimes \mathcal{B}([0,t]) \otimes \mathcal{B}\big(S_d^+\big)$ measurable function $K : \Omega \times \mathbb{R}_+ \times S_d^+ \setminus \{0\} \to \mathbb{R}_+$ with

$$\int_0^t \int_{S_d^+\setminus\{0\}} |K(s,\xi)|\; \nu(ds, d\xi) < \infty \;\; \mathbb{P}\text{-a.s.}$$

for all $t \ge 0$, such that μ^X has the $\mathbb{Q}$-compensator

$$\nu^*(dt, d\xi) := K(t,\xi)\, \nu(dt, d\xi)\,. \quad (3.21)$$

Furthermore, for all $t \ge 0$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\,\Big|_{\mathcal{F}_t} = L_t$$

with

$$\begin{aligned}\log L_t &= \int_0^t \gamma_s\, dW_s - \int_0^t \|\gamma_s\|^2\, ds + \int_0^t \int_{S_d^+\setminus\{0\}} \log K(s,\xi)\, \mu^X(ds, d\xi) \\ &\quad + \int_0^t \int_{S_d^+\setminus\{0\}} (1 - K(s,\xi))\, \nu(ds, d\xi)\,. \quad (3.22)\end{aligned}$$

Definition 3.1. *An* equivalent local martingale measure (ELMM) $\mathbb{Q} \sim \mathbb{P}$ *for the bond market has the property that for all $T > 0$ the discounted bond price process $\frac{P(t,T)}{\beta_t}$, $t \in [0,T]$, is a $\mathbb{Q}$-local martingale.*

Theorem 3.1 (**HJM drift condition on S_d^+**). *A probability measure $\mathbb{Q} \sim \mathbb{P}$ with Radon-Nikodym density* (3.22) *is an* ELMM *if and only if*

$$\begin{aligned}\alpha(t,T) &= -\operatorname{Tr}\Big[\sigma(t,T)\,\Big(b + B(X_t) + 2\sqrt{X_t}\,\gamma_t\, Q\Big)\Big] - 4\operatorname{Tr}\Big[Q\,\sigma(t,T)\, X_s\, \Sigma(t,T)\, Q^\top\Big] \\ &\quad - \int_{S_d^+\setminus\{0\}} \operatorname{Tr}\left[\sigma(t,T)\,\xi\right] e^{\operatorname{Tr}[\Sigma(t,T)\,\xi]} K(t,\xi)\, (m(d\xi) + \operatorname{Tr}[X_s \mu(d\xi)]) \quad (3.23)\end{aligned}$$

for all $T > 0$, $dt \otimes d\mathbb{P}$-a.s.

In this case, the $\mathbb{Q}$-dynamics of the forward rates $f(t,T)$, $0 \le t \le T$, are of the *form*

$$
\begin{aligned}
f(t,T) = f(0,T) + \int_0^t \Big\{ & 4\,\mathrm{Tr}\left[Q\,\sigma(s,T)\,X_s \int_s^T \sigma(s,u)\,du\; Q^\top\right] \\
& - \int_{S_d^+\setminus\{0\}} K(s,\xi)\,\mathrm{Tr}[\sigma(s,T)\,\xi]\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)(m(d\xi)+\mathrm{Tr}[X_s\mu(d\xi)])\Big\}\,ds \\
& + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\left(\mu^X-\nu^*\right)(ds,d\xi) \\
& + 2\int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\right]. \qquad (3.24)
\end{aligned}
$$

Proof. By using (3.20) we see that the discounted bond price process under $\mathbb{Q}$ is

$$
\begin{aligned}
\frac{P(t,T)}{\beta_t} &= P(0,T) + \int_0^t \frac{P(s,T)}{\beta_s} A(s,T)\;ds \\
&\quad + 2\int_0^t \frac{P(s,T)}{\beta_s}\,\mathrm{Tr}\Big[\Sigma(s,T)\sqrt{X_s}\,\gamma_s\,Q\Big]\,ds \\
&\quad + 2\int_0^t \frac{P(s,T)}{\beta_s}\,\mathrm{Tr}\Big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\Big] \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{\beta_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)\left(\mu^X-\nu^*\right)(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{\beta_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)\nu(ds,d\xi) \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{\beta_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)\nu^*(ds,d\xi) \\
&\overset{(3.21)}{=} P(0,T) + \int_0^t \frac{P(s,T)}{\beta_s} A(s,T)\;ds \\
&\quad + 2\int_0^t \frac{P(s,T)}{\beta_s}\,\mathrm{Tr}\Big[\Sigma(s,T)\sqrt{X_s}\,\gamma_s\,Q\Big]\,ds \\
&\quad + 2\int_0^t \frac{P(s,T)}{\beta_s}\,\mathrm{Tr}\Big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\Big] \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{\beta_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)\left(\mu^X-\nu^*\right)(ds,d\xi) \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{\beta_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)(K(s,\xi)-1)\,\nu(ds,d\xi) \qquad (3.25)
\end{aligned}
$$

for all $0 \le t \le T$. Since $\frac{P(t,T)}{\beta_t}$, $t \le T$, has to be a local martingale under $\mathbb{Q}$, the drift in (3.25) must disappear, i.e. for all $0 \le t \le T$

$$
\begin{aligned}
0 &\overset{(2.12)}{=} \int_0^t \frac{P(s,T)}{\beta_s} A(s,T)\;ds + 2\int_0^t \frac{P(s,T)}{\beta_s}\,\mathrm{Tr}\Big[\Sigma(s,T)\sqrt{X_s}\,\gamma_s\,Q\Big]\,ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{\beta_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)(K(s,\xi)-1)\,(m(d\xi)+\mathrm{Tr}[X_s\mu(d\xi)])\,ds.
\end{aligned}
$$

It follows for all $0 \leq t \leq T$ that

$$\begin{aligned} A(t,T) = -2\,\mathrm{Tr}\Big[\Sigma(t,T)\sqrt{X_t}\,\gamma_t\,Q\Big] \\ - \int_{S_d^+\setminus\{0\}} \Big(e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}-1\Big)\,(K(t,\xi)-1)\,(m(d\xi)+\mathrm{Tr}[X_t\mu(d\xi)]) \end{aligned} \tag{3.26}$$

$dt \otimes d\mathbb{P}$-a.s.
Consequently we get for all $0 \leq t \leq T$

$$\begin{aligned} \alpha(t,T) &\overset{(3.4)}{=} -\partial_T A(t,T) - \partial_T F(-\Sigma(t,T)) - \partial_T\,\mathrm{Tr}[R(-\Sigma(t,T))\,X_t] \\ &\underset{(3.17)}{\overset{(3.26)}{=}} -2\,\mathrm{Tr}\Big[\sigma(t,T)\sqrt{X_t}\,\gamma_t\,Q\Big] - \mathrm{Tr}[\sigma(t,T)\,(b+B(X_t))] \\ &\quad - 2\,\mathrm{Tr}\big[Q\,\Sigma(t,T)\,X_t\,\sigma(t,T)\,Q^\top\big] - 2\,\mathrm{Tr}\big[Q\,\sigma(t,T)\,X_t\,\Sigma(t,T)\,Q^\top\big] \\ &\quad - \int_{S_d^+\setminus\{0\}} \mathrm{Tr}\,[\sigma(t,T)\,\xi]\,e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}\,(m(d\xi)+\mathrm{Tr}[X_t\mu(d\xi)]) \\ &\quad - \int_{S_d^+\setminus\{0\}} \mathrm{Tr}\,[\sigma(t,T)\,\xi]\,e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}\,(K(t,\xi)-1)\,(m(d\xi)+\mathrm{Tr}[X_t\mu(d\xi)]) \\ &\overset{(3.11)}{=} -\mathrm{Tr}\Big[\sigma(t,T)\,\Big(b+B(X_t)+2\sqrt{X_t}\,\gamma_t\,Q\Big)\Big] - 4\,\mathrm{Tr}\big[Q\,\sigma(t,T)\,X_t\,\Sigma(t,T)\,Q^\top\big] \\ &\quad - \int_{S_d^+\setminus\{0\}} \mathrm{Tr}\,[\sigma(t,T)\,\xi]\,e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}K(t,\xi)\,(m(d\xi)+\mathrm{Tr}[X_t\mu(d\xi)]) \end{aligned}$$

$dt \otimes d\mathbb{P}$-a.s.
In the calculation we were able to interchange the partial derivative and the integral due to Satz 6.28 in [40] whose prerequisites are fulfilled because of Assumption 1. Hence, $\frac{P(t,T)}{\beta_t}$, $t \leq T$, is a $\mathbb{Q}$-local martingale if and only if equation (3.23) is fulfilled $dt \otimes d\mathbb{P}$-a.s. Equation (3.23) represents the HJM condition on the drift in the affine setting on S_d^+. Then, the forward rate under $\mathbb{Q}$ follows a process of the form

$$\begin{aligned} f(t,T) &\underset{(2.11)}{\overset{(3.1)}{=}} f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \mathrm{Tr}\Big[\sigma(s,T)\,\Big(b+B(X_s)+2\sqrt{X_s}\,\gamma_s\,Q\Big)\Big]\,ds \\ &\quad + \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\;\mu^X(ds,d\xi) + 2\int_0^t \mathrm{Tr}\,\Big[\sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\Big] \\ &\overset{(3.23)}{=} f(0,T) - 4\int_0^t \mathrm{Tr}\,\Big[Q\,\sigma(s,T)\;X_s\,\Sigma(s,T)\;Q^\top\Big]\;ds \\ &\quad - \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}K(s,\xi)\,\nu(ds,d\xi) \\ &\quad + \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\;\mu^X(ds,d\xi) + 2\int_0^t \mathrm{Tr}\,\Big[\sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\Big] \\ &\overset{(3.3)}{=} f(0,T) + 4\int_0^t \mathrm{Tr}\,\Big[Q\,\sigma(s,T)\,X_s\,\int_s^T \sigma(s,u)\,du\;Q^\top\Big]\;ds \\ &\quad + \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\;\Big(\mu^X-\nu^*\Big)(ds,d\xi) \\ &\quad - \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\;\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\Big)\,\nu^*(ds,d\xi) \\ &\quad + 2\int_0^t \mathrm{Tr}\,\Big[\sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\Big] \end{aligned}$$

$$
\begin{aligned}
&\stackrel{(2.12)}{=} f(0,T) + \int_0^t \Bigg\{ 4\,\mathrm{Tr}\left[Q\,\sigma(s,T)\,X_s \int_s^T \sigma(s,u)\,du\; Q^\top \right] \\
&\quad - \int_{S_d^+\setminus\{0\}} K(s,\xi)\,\mathrm{Tr}[\sigma(s,T)\,\xi] \left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1 \right) \left(m(d\xi) + \mathrm{Tr}[X_s \mu(d\xi)] \right) \Bigg\}\, ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi] \left(\mu^X - \nu^* \right)(ds, d\xi) \\
&\quad + 2 \int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\, dW_s^*\, Q \right],
\end{aligned}
$$

where we have used again Proposition 1.28 of Chapter II in [36]. □

Theorem 3.1 shows that the important property of the classical HJM framework, established in [33], that the forward rates are only dependent on the volatility in an arbitrage-free market, still holds in the framework of affine processes on S_d^+.

Next, we want to investigate how the short-rate process $r_t, t \geq 0$, can be represented in the current framework.

Proposition 3.2. *Suppose that $f(0,T)$, $\alpha(t,T)$ and $\sigma(t,T)$ are differentiable in T for all $t \geq 0$, $\partial_T \alpha(t,T)$ is jointly measurable, adapted, and càglàd in t, and $\partial_T \sigma(t,T)$ is jointly measurable, adapted, and càglàd in t. Further, it holds for all $t \geq 0$ that*

$$\int_0^t |\partial_u f(0,u)|\; du < \infty, \tag{3.27}$$

as well as

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |\partial_T \alpha(t,T)|\; dt\, dT < \infty. \tag{3.28}$$

Then, the short-rate process $(r_t)_{t \geq 0}$ is of the form

$$r_t = r_0 + \int_0^t \phi(u)\; du + \int_0^t \mathrm{Tr}[\sigma(u,u)\, dX_u], \tag{3.29}$$

where

$$\phi(u) := \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u)\; ds + \int_0^u \mathrm{Tr}[\partial_u \sigma(s,u)\, dX_s]. \tag{3.30}$$

Proof. We consider representation (3.7) for the short-rate process and investigate the different summands. First, we use (3.27) and see that

$$f(0,t) = r_0 + \int_0^t \partial_u f(0,u)\; du. \tag{3.31}$$

In the following calculations we can use the theorem of Fubini for integrable functions in [40] (Chapter 14, Theorem 14.16) due to the assumption (3.28) and have

$$\int_0^t \alpha(s,t)\; ds = \int_0^t \alpha(s,s)\; ds + \int_0^t \int_0^u \partial_u \alpha(s,u)\; ds\, du, \, t \geq 0. \tag{3.32}$$

Next, we use the stochastic Fubini theorem in [48] (Chapter IV, Theorem 65) since $\partial_T \sigma(t,T)$ is càglàd in t for all $0 \leq t \leq T$ and get

$$
\begin{aligned}
\int_0^t \mathrm{Tr}[\sigma(s,t)\; dX_s] &= \mathrm{Tr}\left[\int_0^t \sigma(s,t)\; dX_s \right] + \mathrm{Tr}\left[\int_0^t \int_s^t \partial_u \sigma(s,u)\; du\, dX_s \right] \\
&= \mathrm{Tr}\left[\int_0^t \sigma(s,t)\; dX_s \right] + \int_0^t \int_0^u \mathrm{Tr}[\partial_u \sigma(s,u)\; dX_s]\; du.
\end{aligned} \tag{3.33}
$$

Putting together (3.31), (3.32), and (3.33), we obtain that the short-rate follows a process of the form (3.29) with ϕ as in (3.30). □

We now calculate the yield process

$$Y(t,T) := -\frac{\log P\,(t,T)}{T-t},\; 0 \le t \le T, \tag{3.34}$$

for $T > 0$ in the HJM framework for affine processes on S_d^+. We recall that the term "yield curve" is used differently in the literature. For example, in [8] it is a combination of simply compounded spot rates for maturities up to one year and annually compounded spot rates for maturities greater than one year. In this paper we will refer to the function $T \mapsto Y(t,T)$ as yield curve in t, see also Section 2.4.4 of [24].

Note that if $f : \mathbb{R}^n \to S_d$ for some $n, d \in \mathbb{N}$, then it is for $a, b \in \mathbb{R}$, $x \in \mathbb{R}^n$:

$$\mathrm{Tr}\left[\int_a^b f(x)^\top\, \partial_x f(x)\; dx\right] = \frac{1}{2}\left(\|f(b)\|^2 - \|f(a)\|^2\right). \tag{3.35}$$

Lemma 3.1. *Let $0 \le t < T$ and let X be an affine process as in* (2.11)*. Under* Assumption 1 *and the* ELMM $\mathbb{Q}$ *the yield for $[t,T]$ can be expressed in the compact form*

$$\begin{aligned}
Y(t,T) = Y(0;t,T) &+ 2\int_0^t \mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}Q^\top\right] ds \\
&+ \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{T-t}\, \nu^*(ds,d\xi) \\
&- \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{T-t}\, \mu^X(ds,d\xi) \\
&- 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}dW_s^* Q\right]
\end{aligned} \tag{3.36}$$

with the continuously compounded forward rate for $[t,T]$ prevailing at 0 *given by*

$$Y(0;t,T) := \frac{1}{T-t}\left(\int_t^T f(0,u)\, du\right) \tag{3.37}$$

and

$$\Gamma(s,t) := \Sigma(s,t)\, X_s \Sigma(s,t) \tag{3.38}$$

for all $s, t \ge 0$.

Proof. Let $0 \le t < T$. Note that for $0 \le s \le t \le T$ it holds

$$\int_t^T \sigma(s,u)\; du \overset{(3.3)}{=} -\left(\Sigma(s,T) - \Sigma(s,t)\right). \tag{3.39}$$

Further, for some $a, b, s \ge 0$ it is

$$\begin{aligned}
&\int_a^b \mathrm{Tr}\big[Q\,\sigma(s,u)\, X_s\, \Sigma(s,u)\; Q^\top\big]\, du \\
&\overset{(3.3)}{=} -\int_a^b \mathrm{Tr}\big[Q\,\partial_u \Sigma(s,u)\, X_s\, \Sigma(s,u)\; Q^\top\big]\, du \\
&= -\int_a^b \mathrm{Tr}\left[\left(Q\,\Sigma(s,u)\sqrt{X_s}\right)^\top \partial_u\left(Q\,\Sigma(s,u)\sqrt{X_s}\right)\right] du \\
&\overset{(3.35)}{=} -\frac{1}{2}\left(\left\|Q\,\Sigma(s,b)\sqrt{X_s}\right\|^2 - \left\|Q\,\Sigma(s,a)\sqrt{X_s}\right\|^2\right) \\
&\overset{(3.38)}{=} -\frac{1}{2}\,\mathrm{Tr}\big[Q\left(\Gamma(s,b) - \Gamma(s,a)\right) Q^\top\big].
\end{aligned} \tag{3.40}$$

Then, the yield for $[t,T]$ is

$$
\begin{aligned}
Y(t,T) &= \frac{1}{T-t}\left(\int_t^T f(t,u)\,du\right) \\
&\overset{(3.24)}{=} \int_t^T \frac{f(0,u)}{T-t}\,du - \frac{4}{T-t}\int_t^T\int_0^t \mathrm{Tr}\big[Q\,\sigma(s,u)\,X_s\Sigma(s,u)\,Q^\top\big]\,ds\,du \\
&\quad + \frac{1}{T-t}\int_t^T\int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,u)\,\xi]\,\big(\mu^X-\nu^*\big)(ds,d\xi)\ du \\
&\quad - \frac{1}{T-t}\int_t^T\int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\Big)\,\nu^*(ds,d\xi)\ du \\
&\quad + \frac{2}{T-t}\int_t^T\int_0^t \mathrm{Tr}\Big[\sigma(s,u)\,\sqrt{X_s}\,dW_s^*Q\Big]\,du \\
&\overset{(3.37)}{=} Y(0;t,T) - \frac{4}{T-t}\int_0^t\int_t^T \mathrm{Tr}\big[Q\,\sigma(s,u)\,X_s\Sigma(s,u)\,Q^\top\big]\,du\,ds \\
&\quad - \frac{1}{T-t}\int_0^t\int_t^T\int_{S_d^+\setminus\{0\}} \partial_u\,\mathrm{Tr}[\Sigma(s,u)\,\xi]\,\big(\mu^X-\nu^*\big)(du,d\xi)\ ds \\
&\quad + \frac{1}{T-t}\int_0^t\int_t^T\int_{S_d^+\setminus\{0\}} \partial_u e^{\mathrm{Tr}[\Sigma(s,u)\,\xi]}\,\nu^*(du,d\xi)\ ds \\
&\quad - \frac{1}{T-t}\int_0^t\int_t^T\int_{S_d^+\setminus\{0\}} \partial_u\,\mathrm{Tr}[\Sigma(s,u)\,\xi]\,\nu^*(du,d\xi)\ ds \\
&\quad + \frac{2}{T-t}\int_0^t \mathrm{Tr}\left[\int_t^T \sigma(s,u)\,du\,\sqrt{X_s}\,dW_s^*Q\right] \qquad (3.41) \\
&\overset{(3.39)}{\underset{(3.40)}{=}} Y(0;t,T) + 2\int_0^t \mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}Q^\top\right]ds \\
&\quad + \int_0^t\int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{T-t}\,\nu^*(ds,d\xi) \\
&\quad - \int_0^t\int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{T-t}\,\mu^X(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}\,dW_s^*Q\right].
\end{aligned}
$$

At (3.41) we used the Fubini theorem for integrable functions (cf. [40], Chapter 14, Theorem 14.16) and the stochastic Fubini theorem (cf. [48], Chapter IV, Theorem 65). These theorems can be used because of Assumption 1. □

Corollary 3.2. *By* (2.12), (3.21), *and* (3.36) *we obtain that*

$$
\begin{aligned}
Y(t,T) = Y(0;t,T) + \int_0^t &\left\{2\,\mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}Q^\top\right]\right. \\
&\left.+\int_{S_d^+\setminus\{0\}} \frac{M(s,t,T,\xi)\,K(s,\xi)}{T-t}\,(m(d\xi)+\mathrm{Tr}[X_s\mu(d\xi)])\right\}ds \\
&- \int_0^t\int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{T-t}\,\big(\mu^X-\nu^*\big)(ds,d\xi) \\
&- 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}dW_s^*Q\right] \qquad (3.42)
\end{aligned}
$$

with

$$M(s,t,T,\xi) := e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\xi]} - \mathrm{Tr}\left[\left(\Sigma(s,T) - \Sigma(s,t)\right)\xi\right]. \tag{3.43}$$

Proof. Let $0 \le t < T$. Then, we have

$$\begin{aligned}
Y(t,T) &\overset{(3.36)}{=} Y(0;t,T) + 2\int_0^t \mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}Q^\top\right] ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{T-t}\, \nu^*(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[\left(\Sigma(s,T)-\Sigma(s,t)\right)\xi\right]}{T-t}\, \mu^X(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}dW_s^* Q\right] \\
&= Y(0;t,T) + 2\int_0^t \mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}Q^\top\right] ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]} - \mathrm{Tr}\left[\left(\Sigma(s,T)-\Sigma(s,t)\right)\xi\right]}{T-t}\, \nu^*(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[\left(\Sigma(s,T)-\Sigma(s,t)\right)\xi\right]}{T-t} \left(\mu^X - \nu^*\right)(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}dW_s^* Q\right] \\
&\underset{(3.43)}{\overset{(2.12)}{=}} Y(0;t,T) + \int_0^t \Bigg\{ 2\,\mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}Q^\top\right] \\
&\quad + \int_{S_d^+\setminus\{0\}} \frac{M(s,t,T,\xi)\,K(s,\xi)}{T-t}\left(m(d\xi) + \mathrm{Tr}[X_s\mu(d\xi)]\right)\Bigg\} ds \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[\left(\Sigma(s,T)-\Sigma(s,t)\right)\xi\right]}{T-t} \left(\mu^X - \nu^*\right)(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}dW_s^* Q\right].
\end{aligned}$$

□

4. Long-Term Yield in an Affine HJM Setting

The expression "long-term yield" is subject to different interpretations in the literature. For instance, the European Central Bank understands the market yields of government bonds with time to maturity close to 10 years as long-term interest rates (cf. [23]), whereas in [51] also high-grade bonds with time to maturity longer than 20 years are examined to investigate long-term yields. In [53] it is pointed out that for the valuation of some financial securities yield curves with maturities up to 100 years are necessary. Here we interpret "long-term yield" as the yield with time to maturity going to infinity. This approach, adopted by [6], [10], [22], [38], is useful for modelling interest rates within a long-time horizon because the asymptotic behaviour can give information about the shape of the yield curve in the long run where only few empirical data is available. Here we study the asymptotic behaviour of the long-term yield in the affine HJM setting, introduced in Section 3.

Throughout this section in the setting outlined in Section 2 we assume directly that $\mathbb{P}$ is an ELMM for $\frac{P(t,T)}{\beta_t}$, $t \in [0,T]$, for all $T > 0$. More precisely, X is a conservative affine process on S_d^+, $d \ge 2$, with representation (2.11), (2.12) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ and the yield takes the form (3.42), where we write ν instead of ν^* for the sake of simplicity.

Assumption 2. *Let $\Sigma(s,t)$ be defined as in (3.3) for all $0 \leq s \leq t$ and W a matrix variate Brownian motion. There exists a progressively measurable process $w \in L(W)$ with values in S_d^+ such that for every $i, j \in \{1, \ldots, d\}$, w_{ij} is a càdlàg process with*

$$\frac{1}{\sqrt{t}} \left| \Sigma(s,t)_{ij} \right| \leq w_{ij}(s) \ \ \mathbb{P}\text{-a.s.} \tag{4.1}$$

for all $0 \leq s \leq t$ and $t \neq 0$.

Definition 4.1. *The* long-term yield $(\ell_t)_{t \geq 0}$ *is the process defined by*

$$\ell_t := \lim_{T \to \infty} Y(t,T)\,, \tag{4.2}$$

where $Y(t,T)\,, t \in [0,T]$, is the yield process for $T \geq 0$ given by equation (3.34).

Definition 4.2. *If the forward rate process is defined as in (3.1), the* long-term drift $\mu_\infty(t)\,, t \geq 0$, *is the process on $\mathcal{M}_d$ given by*

$$\mu_\infty(t) := \lim_{T \to \infty} \frac{\Gamma(t,T)}{T-t} = \lim_{T \to \infty} \frac{\Gamma(t,T)}{T} \ \ \mathbb{P}\text{-a.s.} \tag{4.3}$$

for all $t \geq 0$, where $\Gamma(t,T)\,, t \in [0,T]$, is introduced in (3.38) for every $T \geq 0$. Furthermore, the long-term volatility $\sigma_\infty(t)\,, t \geq 0$, *is the process on $\mathcal{M}_d$ given by*

$$\sigma_\infty(t) := \lim_{T \to \infty} \frac{\Sigma(t,T)}{T-t} = \lim_{T \to \infty} \frac{\Sigma(t,T)}{T} \ \ \mathbb{P}\text{-a.s.} \tag{4.4}$$

for all $t \geq 0$, where $\Sigma(t,T)\,, t \in [0,T]$, is introduced in (3.3) for every $T \geq 0$.

Here we are supposing that the limits (4.2), (4.3) and (4.4) are well-defined. The long-term yield can be characterised as an integral of μ_∞ and σ_∞ by using the following results.

Proposition 4.1. *Let $0 \leq t \leq T$. The* long-term yield *at 0 is*

$$\lim_{T \to \infty} Y(0;t,T) = \lim_{T \to \infty} Y(0,T) = \ell_0 \ \ \mathbb{P}\text{-a.s.}$$

Proof. Cf. Proposition 3.3 of [6]. □

Proposition 4.2. *Under* Assumption 1 *and* 2*, it holds for all $t \geq 0$:*

$$\lim_{T \to \infty} 2 \int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T) - \Sigma(s,t)}{T-t} \sqrt{X_s}\, dW_s\, Q \right] = 2 \int_0^t \mathrm{Tr}\left[\sigma_\infty(s) \sqrt{X_s}\, dW_s\, Q \right], \tag{4.5}$$

where $\sigma_\infty(s)\,, s \geq 0$, is the long-term volatility process defined by equation (4.4), $\Sigma(s,t)$, $s \geq 0$, is defined for all $t \geq 0$ as in (3.3), and the convergence in (4.5) is uniform on compacts in probability (ucp).

Proof. Fix $t \geq 0$. By Assumption 1 we have that for all compact intervals $[a,b]$ with $0 \leq a < b$

$$\sup_{t \in [a,b]} \left| \int_0^t \mathrm{Tr}\left[2\, Q\, \Sigma(s,t) \sqrt{X_s}\, dW_s \right] \right| < \infty \ \ \mathbb{P}\text{-a.s.}$$

Consequently on every compact interval $[a,b]$

$$\frac{1}{T} \sup_{t \in [a,b]} \int_0^t \mathrm{Tr}\left[2\, Q\, \Sigma(s,t) \sqrt{X_s}\, dW_s \right] \overset{T \to \infty}{\longrightarrow} 0 \ \ \mathbb{P}\text{-a.s.}$$

Therefore

$$\frac{1}{T} \int_0^t \mathrm{Tr}\left[2\, Q\, \Sigma(s,t) \sqrt{X_s}\, dW_s \right] \overset{T \to \infty}{\longrightarrow} 0 \text{ in ucp.} \tag{4.6}$$

Next, we define $H^T := H_s^T, s \geq 0$, with

$$H_s^T := 2\frac{Q\,\Sigma(s,T)\,\sqrt{X_s}}{T}\,. \tag{4.7}$$

Then for $T \to \infty:\ H_s^T \to 2\,Q\,\sigma_\infty(s)\,\sqrt{X_s}$ a.s. for all $s \geq 0$.
Since we investigate long-term interest rates it is sufficient to impose long times of maturity, say $T \geq 1$. Due to Assumption 2, we then have that for all $0 \leq s \leq T$ with $T \geq 1$

$$\begin{aligned}
\left\|H_s^T\right\| &\overset{(4.7)}{=} \frac{2}{T}\left\|Q\,\Sigma(s,T)\,\sqrt{X_s}\right\| \\
&= \frac{2}{T}\left(\mathrm{Tr}\left[\sqrt{X_s}\,\Sigma(s,T)\,Q^\top Q\,\Sigma(s,T)\,\sqrt{X_s}\right]\right)^{1/2} \\
&= \frac{2}{T}\left(\sum_{i,j,k,l,m,n}\sqrt{X_{ij,s}}\,\Sigma(s,T)_{jk}\,Q_{kl}^\top\,Q_{lm}\,\Sigma(s,T)_{mn}\,\sqrt{X_{ni,s}}\right)^{1/2} \\
&\overset{(4.1)}{\leq} \frac{2}{T}\left(\sum_{i,j,k,l,m,n}\sqrt{T}\,\sqrt{X_{ij,s}}\,w_{jk}(s)\;Q_{kl}^\top\,Q_{lm}\,\sqrt{T}\,w_{mn}(s)\;\sqrt{X_{ni,s}}\right)^{1/2} \\
&= \frac{2}{\sqrt{T}}\left(\sum_{i,j,k,l,m,n}\sqrt{X_{ij,s}}\,w_{jk}(s)\;Q_{kl}^\top\,Q_{lm}\,w_{mn}(s)\;\sqrt{X_{ni,s}}\right)^{1/2} \\
&\leq 2\left(\sum_{i,j,k,l,m,n}\sqrt{X_{ij,s}}\,w_{jk}(s)\;Q_{kl}^\top\,Q_{lm}\,w_{mn}(s)\;\sqrt{X_{ni,s}}\right)^{1/2} \\
&= 2\left(\mathrm{Tr}\left[\sqrt{X_s}\,w(s)\;Q^\top Q\,w(s)\,\sqrt{X_s}\right]\right)^{1/2} \\
&= 2\left\|Q\,w(s)\,\sqrt{X_s}\right\| =: h(s)\,.
\end{aligned}$$

Further, it is $w \in L(W)$ due to Assumption 2 and we know from Theorem 2.2 that $\sqrt{X} \in L(W)$. By using Theorem 16 in Chapter IV, Section 2 of [48] it follows $h \in L(W)$. Then, applying the dominated convergence theorem for semimartingales (cf. Theorem 32 in Chapter IV, Section 2 of [48]), we get:

$$\int_0^t \mathrm{Tr}\left[2\,\frac{Q\,\Sigma(s,T)\,\sqrt{X_s}}{T}\,dW_s\right] \overset{T\to\infty}{\longrightarrow} 2\int_0^t \mathrm{Tr}\left[\sigma_\infty(s)\,\sqrt{X_s}\,dW_s\,Q\right] \text{ in ucp.} \tag{4.8}$$

It follows due to Lemma 5.8 of [25], (4.6), and (4.8):

$$2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{T-t}\sqrt{X_s}dW_sQ\right] \overset{T\to\infty}{\longrightarrow} 2\int_0^t \mathrm{Tr}\left[\sigma_\infty(s)\,\sqrt{X_s}\,dW_s\,Q\right] \text{ in ucp.}$$

□

Proposition 4.3. *Under* Assumption 1 *and* 2, *it holds for all $t \geq 0$:*

$$\lim_{T\to\infty} 2\int_0^t \mathrm{Tr}\left[Q\,\frac{\Gamma(s,T)-\Gamma(s,t)}{T-t}\,Q^\top\right]ds = 2\int_0^t \mathrm{Tr}\left[Q\,\mu_\infty(s)\,Q^\top\right]ds, \tag{4.9}$$

where $\mu_\infty(s)\,, s \geq 0$, is the long-term drift process defined by equation (4.3), $\Gamma(s,t)$, *$s \geq 0$, is defined for all $t \geq 0$ as in* (3.38), *and the convergence in* (4.9) *is in ucp.*

Proof. Fix $t \geq 0$. Since the process $\Gamma(s,t)\,, s,t \geq 0$, is continuous in t for all fixed s and càdlàg in s for all fixed t it follows by (4) of Section 2.8 in [2] that $\Gamma(s,t)\,, s \geq 0$, is bounded on all compact intervals $[a,b]$ with $t \in [a,b]$ and $0 \le a < b$ for a.e. $\omega \in \Omega$, i.e.

$$\sup_{t\in[a,b]} \left| \int_0^t \mathrm{Tr}\big[Q\,\Gamma(s,t)\,Q^\top\big]\,ds \right| < \infty \ \ \mathbb{P}\text{-a.s.}$$

Consequently on every compact interval $[a,b]$

$$\frac{1}{T} \sup_{t\in[a,b]} \int_0^t \mathrm{Tr}\big[Q\,\Gamma(s,t)\,Q^\top\big]\,ds \overset{T\to\infty}{\longrightarrow} 0 \ \ \mathbb{P}\text{-a.s.}$$

Therefore

$$\frac{1}{T} \int_0^t \mathrm{Tr}\big[Q\,\Gamma(s,t)\,Q^\top\big]\,ds \overset{T\to\infty}{\longrightarrow} 0 \ \text{ in ucp.} \tag{4.10}$$

Let us define $G^T := G^T_s, s \ge 0$, with

$$G^T_s := 2\frac{Q\,\Gamma(s,T)\,Q^\top}{T}\,. \tag{4.11}$$

Then for $T \to \infty:\ G^T_s \to 2\,Q\,\mu_\infty(s)\,Q^\top$ a.s. for all $s \ge 0$.
By Assumption 2 we have that for all $i,j \in \{1,\dots,d\}$ and $0 \le s \le T$:

$$\begin{aligned}
\Gamma(s,T)_{ij} &\overset{(3.38)}{=} (\Sigma(s,T)\,X_s\Sigma(s,T))_{ij} \ = \ \sum_{k,l} \Sigma(s,T)_{ik}\,X_{kl,s}\Sigma(s,T)_{lj} \\
&\overset{(4.1)}{\le} \sum_{k,l} \sqrt{T}\,w_{ik}(s)\,X_{kl,s}\,\sqrt{T}\,w_{lj}(s) \ = \ T\,(w(s)\,X_s w(s))_{ij}\,.
\end{aligned} \tag{4.12}$$

Therefore we have that for all $0 \le s \le T$

$$\begin{aligned}
\left\|G^T_s\right\| &\overset{(4.11)}{=} \frac{2}{T}\left\|Q\,\Gamma(s,T)\,Q^\top\right\| \\
&= \frac{2}{T}\left(\mathrm{Tr}\big[Q\,\Gamma(s,T)\,Q^\top Q\,\Gamma(s,T)\,Q^\top\big]\right)^{1/2} \\
&= \frac{2}{T}\left(\sum_{i,j,k,l,m,n} Q_{ij}\,\Gamma(s,T)_{jk}\,Q^\top_{kl}\,Q_{lm}\,\Gamma(s,T)_{mn}\,Q^\top_{ni}\right)^{1/2} \\
&\overset{(4.12)}{\le} \frac{2}{T}\left(\sum_{i,j,k,l,m,n} Q_{ij}\,T\,(w(s)\,X_s w(s))_{jk}\,Q^\top_{kl}\,Q_{lm}\,T\,(w(s)\,X_s w(s))_{mn}\,Q^\top_{ni}\right)^{1/2} \\
&= 2\left(\mathrm{Tr}\big[Q\,w(s)\,X_s w(s)\,Q^\top Q\,w(s)\,X_s w(s)\,Q^\top\big]\right)^{1/2} \\
&= 2\left\|Q\,w(s)\,X_s w(s)\,Q^\top\right\| =: g(s)\,,
\end{aligned}$$

where g is a càdlàg process. It follows $\int_0^t g(s)\,ds < \infty$ for all $t \ge 0$ by (4) of Section 2.8 in [2] and we can apply the DCT for progressive processes (cf. Corollary 6.26 in Chapter 6 of [40]). By using Lemma 5.8 of [25] and (4.10) we obtain (4.9). □

Proposition 4.4. *Under* Assumption 1 *and* 2*, it holds for all* $t \ge 0$*:*

$$\int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}[(\Sigma(s,T) - \Sigma(s,t))\,\xi]}{T-t}\,\mu^X(ds,d\xi) \overset{T\to\infty}{\longrightarrow} \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\,, \tag{4.13}$$

where $\sigma_\infty(s)\,, s \ge 0$*, is the long-term volatility process defined by equation* (4.4)*,* $\Sigma(s,t)$*,* $s \ge 0$*, is defined for all* $t \ge 0$ *as in* (3.3)*, and the convergence in* (4.13) *is in ucp.*

Proof. Fix $t \geq 0$. First, notice that

$$\begin{aligned}
\left| \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\Sigma(s,t)\,\xi]\,\mu^X(ds,d\xi) \right| &\leq \int_0^t \int_{S_d^+} |\mathrm{Tr}[\Sigma(s,t)\,\xi]|\,\mu^X(ds,d\xi) \\
&\leq \sqrt{t} \int_0^t \int_{S_d^+ \setminus \{0\}} \frac{1}{\sqrt{t}} \|\Sigma(s,t)\|\,\|\xi\|\,\mu^X(ds,d\xi) \\
&\overset{(4.1)}{\leq} \sqrt{t} \int_0^t \int_{S_d^+ \setminus \{0\}} \|w(s)\|\,\|\xi\|\,\mu^X(ds,d\xi) \\
&\leq \sqrt{t} \sup_{u \in [0,t]} \|w(u)\| \int_0^t \int_{S_d^+ \setminus \{0\}} \|\xi\|\ \mu^X(ds,d\xi)\ .
\end{aligned}$$

Define $q(t) := \sqrt{t}\,\sup_{u \in [0,t]} \|w(u)\|\, Z_t$ with

$$Z_t := \int_0^t \int_{S_d^+ \setminus \{0\}} \|\xi\|\ \mu^X(ds,d\xi)\,, t \geq 0\,.$$

Note that $Z_t, t \geq 0$, is a well-defined càdlàg process by (2.10). Then for all compact intervals $[a,b]$ with $0 \leq a < b$ it is

$$\begin{aligned}
\sup_{t \in [a,b]} |q(t)| = \sup_{t \in [a,b]} \left| \sqrt{t} \sup_{u \in [0,t]} \|w(u)\|\, Z_t \right| &\leq \sqrt{b} \sup_{t \in [a,b]} \sup_{u \in [0,t]} \|w(u)\| \sup_{t \in [a,b]} Z_t \\
= \sqrt{b} \sup_{u \in [0,b]} \|w(u)\| \sup_{t \in [a,b]} Z_t < \infty &
\end{aligned}$$

because of (4) of Section 2.8 in [2] applied for the càdlàg processes $\|w(t)\|\,, t \geq 0$, and $Z_t, t \geq 0$. Consequently on every compact interval $[a,b]$

$$\frac{1}{T} \sup_{t \in [a,b]} \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\Sigma(s,t)\,\xi]\,\mu^X(ds,d\xi) \overset{T \to \infty}{\longrightarrow} 0\ \ \mathbb{P}\text{-a.s.}$$

Therefore

$$\frac{1}{T} \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\Sigma(s,t)\,\xi]\,\mu^X(ds,d\xi) \overset{T \to \infty}{\longrightarrow} 0\ \text{ in ucp.} \tag{4.14}$$

Due to Assumption 2 we have for $s, t \leq T$ and $T \geq 1$ that

$$\frac{|\mathrm{Tr}\,[\Sigma(s,T)\,\xi]|}{T} \leq \frac{1}{T} \|\Sigma(s,T)\|\,\|\xi\| \overset{(4.1)}{\leq} \frac{1}{\sqrt{T}} \|w(s)\|\,\|\xi\| \leq \|w(s)\|\,\|\xi\| =: j(s,\xi)\,.$$

We first show that the process j is integrable with respect to the random measure μ^X on $[0,t] \times S_d^+ \setminus \{0\}$ for all $t \geq 0$:

$$\begin{aligned}
\int_0^t \int_{S_d^+ \setminus \{0\}} j(s,\xi)\,\mu^X(ds,d\xi) &= \int_0^t \int_{S_d^+ \setminus \{0\}} \|w(s)\|\,\|\xi\|\ \mu^X(ds,d\xi) \\
&\leq \sup_{u \in [0,t]} \|w(u)\|\, Z_t < \infty
\end{aligned}$$

due to (4) of Section 2.8 in [2] applied for the càdlàg process $\|w(t)\|\,, t \geq 0$, and by (2.10). We have with the DCT and (4.4) that for all fixed $t \geq 0$

$$\int_0^t \int_{S_d^+ \setminus \{0\}} \frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T} \mu^X(ds,d\xi) \overset{T \to \infty}{\longrightarrow} \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\ \ \mathbb{P}\text{-a.s.} \tag{4.15}$$

Then, by (4.15) applied for $t = b$ and by (4.17) from Lemma 4.1, it follows that

$$\sup_{t \in [a,b]} \int_0^t \int_{S_d^+ \setminus \{0\}} \frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T} \mu^X(ds,d\xi) \overset{T \to \infty}{\longrightarrow} \sup_{t \in [a,b]} \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)$$

$\mathbb{P}$-a.s. and therefore in probability. Hence by [48], page 57

$$\lim_{T\to\infty}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}\left[\Sigma(s,T)\,\xi\right]}{T}\,\mu^X(ds,d\xi)=\int_0^t\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\quad\text{in ucp.}\tag{4.16}$$

By using (4.14) as well as (4.16) it follows (4.13). □

Lemma 4.1. *With* $\Sigma(t,T)\,,t\geq 0,$ *defined as in* (3.3) *and* $\sigma_\infty(t)\,,t\geq 0,$ *as in* (4.4) *it holds for* $0\leq a<b$

$$\begin{aligned}\Bigg|\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\mu^X(ds,d\xi)-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\Bigg|\\ \leq\int_0^b\int_{S_d^+\setminus\{0\}}\left|\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right|\mu^X(ds,d\xi)\,.\end{aligned}\tag{4.17}$$

Proof. Let $0\leq a\leq b$. Then, we have

$$\begin{aligned}&\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\mu^X(ds,d\xi)-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\\ &\quad=\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]+\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\\ &\qquad-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\\ &\quad\leq\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\,.\end{aligned}\tag{4.18}$$

Furthermore

$$\begin{aligned}&\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\mu^X(ds,d\xi)-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\\ &\quad=\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\mu^X(ds,d\xi)\\ &\qquad-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\mathrm{Tr}[\sigma_\infty(s)\,\xi]-\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}+\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\right)\mu^X(ds,d\xi)\\ &\quad\geq-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\mathrm{Tr}[\sigma_\infty(s)\,\xi]-\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\right)\mu^X(ds,d\xi)\\ &\quad=\inf_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\,.\end{aligned}\tag{4.19}$$

Hence, it follows from (4.18) and (4.19) that

$$\begin{aligned}&\Bigg|\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}\mu^X(ds,d\xi)-\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\Bigg|\\ &\quad\leq\left|\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\right|\\ &\qquad\vee\left|\inf_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\right|\\ &\quad\leq\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\left|\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{T}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right|\mu^X(ds,d\xi)\,.\end{aligned}$$

□

Proposition 4.5. *Under* Assumption 1 *and* 2*, it holds for all* $t \geq 0$*:*

$$\lim_{T\to\infty} \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{T-t}\, \nu(ds, d\xi) = 0\,, \tag{4.20}$$

where $\Gamma(s,t)$*,* $s \geq 0$*, is defined for all* $t \geq 0$ *as in* (3.38)*, and the convergence in* (4.20) *is in ucp.*

Proof. Fix $t \geq 0$. We note that the left-hand side of (4.20) is equal to

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^t \int_{S_d^+\setminus\{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}\right) \nu(ds,d\xi) - \int_0^t \int_{S_d^+\setminus\{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}\right) \nu(ds,d\xi)\right). \tag{4.21}$$

Hence we study the limit (4.21) and we introduce for all $u \in S_d^+$

$$\tilde{F}(u) := \int_{S_d^+\setminus\{0\}} \left(e^{-\mathrm{Tr}[u\xi]} - 1\right) m(d\xi)\ , \tag{4.22}$$

$$\tilde{R}(u) := \int_{S_d^+\setminus\{0\}} \left(e^{-\mathrm{Tr}[u\xi]} - 1\right) \mu(d\xi)\ . \tag{4.23}$$

Then, we can write due to (2.12), (4.22), and (4.23) that

$$\int_0^t \int_{S_d^+\setminus\{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\xi]}\right) \nu(ds,d\xi) = -\int_0^t \left(\tilde{F}(-\Sigma(s,t)) + \mathrm{Tr}\left[\tilde{R}(-\Sigma(s,t))\, X_s\right]\right) ds.$$

The process $\tilde{F}(-\Sigma(s,t)) + \mathrm{Tr}\left[\tilde{R}(-\Sigma(s,t))\, X_s\right], s \in [0,t]$, is càdlàg for all $t \geq 0$ since $\tilde{F}$ and $\tilde{R}$ are continuous functions and X is càdlàg. Due to this and (4) of Section 2.8 in [2] we get that for all compact intervals $[a,b]$ with $a,b \geq 0$

$$\sup_{t\in[a,b]} \int_0^t \int_{S_d^+\setminus\{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}\right) \nu(ds,d\xi) < \infty \ \ \mathbb{P}\text{-a.s.}$$

Consequently on every compact interval $[a,b]$

$$\frac{1}{T} \sup_{t\in[a,b]} \int_0^t \int_{S_d^+\setminus\{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}\right) \nu(ds,d\xi) \overset{T\to\infty}{\longrightarrow} 0 \ \ \mathbb{P}\text{-a.s.}$$

Therefore

$$\frac{1}{T} \int_0^t \int_{S_d^+\setminus\{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}\right) \nu(ds,d\xi) \overset{T\to\infty}{\longrightarrow} 0 \ \text{ in ucp.} \tag{4.24}$$

Next, we use the inequality

$$1 - e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} = 1 - e^{-\mathrm{Tr}[-\Sigma(s,T)\,\xi]} \leq 1 \wedge \mathrm{Tr}[-\Sigma(s,T)\,\xi] \overset{(4.1)}{\leq} 1 \wedge \sqrt{T}\, \mathrm{Tr}[w(s)\,\xi] \tag{4.25}$$

which holds for all $\xi \in S_d^+$ and a.e. $\omega \in \Omega$, to see that for all $0 \leq s \leq T$ with $T \geq 1$, and for a.e. $\omega \in \Omega$

$$\frac{1 - e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}}{T} \overset{(4.25)}{\leq} \frac{1}{T} \wedge \frac{1}{\sqrt{T}} \mathrm{Tr}[w(s)\,\xi] \leq 1 \wedge \mathrm{Tr}[w(s)\,\xi] \leq 1 \wedge \|w(s)\|\, \|\xi\| =: i(s,\xi)\ .$$

Since we investigate long-term interest rates it is sufficient to impose long times of maturity, say $T \geq 1$.

Then, we show that the process i is integrable with respect to the random measure ν on $[0,t]\times S_d^+$:

$$\begin{aligned}
\int_0^t\int_{S_d^+\setminus\{0\}} i(s,\xi)\,\nu(ds,d\xi) &= \int_0^t\int_{S_d^+\setminus\{0\}} i(s,\xi)\left(\mathbb{1}_{\{\|w(s)\|\le 1\}}+\mathbb{1}_{\{\|w(s)\|>1\}}\right)\nu(ds,d\xi)\\
&= \int_0^t\int_{S_d^+\setminus\{0\}} i(s,\xi)\,\mathbb{1}_{\{\|w(s)\|\le 1\}}\,\nu(ds,d\xi)\\
&\quad+\int_0^t\int_{S_d^+\setminus\{0\}} i(s,\xi)\,\mathbb{1}_{\{\|w(s)\|>1\}}\,\nu(ds,d\xi)\\
&\le \int_0^t\int_{S_d^+\setminus\{0\}} \mathbb{1}_{\{\|w(s)\|\le 1\}}\,(1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\quad+\int_0^t\int_{S_d^+\setminus\{0\}} \mathbb{1}_{\{\|w(s)\|>1\}}\,\|w(s)\|\,(1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\overset{(2.12)}{\le} \int_0^t\int_{S_d^+\setminus\{0\}} (1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\quad+\int_0^t \|w(s)\|\,\mathbb{1}_{\{\|w(s)\|>1\}}\,ds\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,m(d\xi)\\
&\quad+\mathrm{Tr}\left[\int_0^t \|w(s)\|\,\mathbb{1}_{\{\|w(s)\|>1\}}X_s\,ds\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,\mu(d\xi)\right]\\
&\le \int_0^t\int_{S_d^+\setminus\{0\}} (1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\quad+\int_0^t \|w(s)\|\;ds\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,m(d\xi)\\
&\quad+\mathrm{Tr}\left[\int_0^t \|w(s)\|\,X_s\,ds\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,\mu(d\xi)\right]\\
&< \infty\ \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

because of (2.4), (2.5), and (4) of Section 2.8 in [2] applied for the càdlàg processes X and $\|w(t)\|\,X_t, t\ge 0$.
Then by the DCT we have that for all $t\ge 0$

$$\int_0^t\int_{S_d^+\setminus\{0\}}\frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1}{T}\,\nu(ds,d\xi)\overset{T\to\infty}{\longrightarrow}0\ \ \mathbb{P}\text{-a.s.}\tag{4.26}$$

With the same argument as in Proposition 4.4, we then obtain that for $0\le a<b$

$$\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{1-e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}}{T}\,\nu(ds,d\xi)\le\int_0^b\int_{S_d^+\setminus\{0\}}\frac{1-e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}}{T}\nu(ds,d\xi)\,,$$

since ν is given by (2.12).
This converges to 0 by (4.26) applied for $t=b$. Therefore a.e. $\omega\in\Omega$

$$\sup_{t\in[a,b]}\int_0^t\int_{S_d^+\setminus\{0\}}\frac{1-e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}}{T}\,\nu(ds,d\xi)\overset{T\to\infty}{\longrightarrow}0\,,\tag{4.27}$$

i. e. by [48], page 57

$$\int_0^t\int_{S_d^+\setminus\{0\}}\frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1}{T}\,\nu(ds,d\xi)\overset{T\to\infty}{\longrightarrow}0\ \text{ in ucp.}\tag{4.28}$$

The result (4.20) follows then by (4.21), (4.24), and (4.28). $\square$

By Lemma 3.1, Proposition 4.1, Proposition 4.2, Proposition 4.3, Proposition 4.4, and Proposition 4.5, the long-term yield can be written in the following way:

$$\begin{aligned}\ell_t = \ell_0 + 2\int_0^t \mathrm{Tr}\big[Q\,\mu_\infty(s)\,Q^\top\big]\;ds - 2\int_0^t \mathrm{Tr}\Big[\sigma_\infty(s)\,\sqrt{X_s}\,dW_s\,Q\Big]\\ - \int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\,,\; t\geq 0,\end{aligned} \tag{4.29}$$

whereas the convergence is uniformly on compacts in probability.

Next, we want to closer investigate equation (4.29).

Lemma 4.2. *Under the setting outlined in* Section 3 *the long-term volatility has to vanish so that the long-term yield exists, i.e. for all* $t \geq 0$*:*

$$\ell_t < \infty \Rightarrow \sigma_\infty(t) = 0\,.$$

Proof. Let $0 \leq t \leq T$. We assume that $0 < \|\sigma_\infty(t)\| < \infty$. It follows from (4.4) that for all $i,j \in \{1,\dots,d\}$ we have $\Sigma(t,T)_{ij} \in \mathrm{O}(T-t)$. Then we get for all $t \geq 0$:

$$\begin{aligned}\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] &\overset{(4.3)}{=} \sum_{i,j,k} Q_{ij}\lim_{T\to\infty}\frac{\Gamma(t,T)_{jk}}{T-t}\,Q^\top_{ki}\\ &\overset{(3.38)}{=} \lim_{T\to\infty}\frac{1}{T-t}\sum_{i,j,k,l,m} Q_{ij}\,\Sigma(t,T)_{jl}\,X_{lm,t}\,\Sigma(t,T)_{mk}\,Q_{ik}\\ &= \infty\;\;\mathbb{P}\text{-a.s.}\end{aligned}$$

That is a contradiction to the existence of the long-term yield. □

Lemma 4.3. *Under the setting outlined in* Section 3 *the trace of the long-term drift is a non-negative process, i.e. for all* $t \geq 0$*:*

$$\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] \geq 0\,.$$

Proof. Let $t \geq 0$. Then

$$\begin{aligned}\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] &\overset{(4.3)}{=} \lim_{T\to\infty}\frac{1}{T-t}\,\mathrm{Tr}\big[Q\,\Gamma(t,T)\,Q^\top\big]\\ &\overset{(3.38)}{=} \lim_{T\to\infty}\frac{1}{T-t}\,\mathrm{Tr}\big[Q\,\Sigma(t,T)\,X_t\,\Sigma(t,T)\,Q^\top\big]\\ &= \lim_{T\to\infty}\frac{1}{T-t}\,\Big\|\sqrt{X_t}\,\Sigma(t,T)\,Q^\top\Big\|^2 \geq 0\;\;\mathbb{P}\text{-a.s.}\end{aligned} \tag{4.30}$$

□

By Lemmas 4.2 and 4.3 we get that if the long-term yield process $(\ell_t)_{t\geq 0}$ exists and is finite, then it has the form

$$\ell_t = \ell_0 + 2\int_0^t \mathrm{Tr}\big[Q\,\mu_\infty(s)\,Q^\top\big]\,ds,\; t \geq 0, \tag{4.31}$$

with $\mathrm{Tr}\big[Q\,\mu_\infty(s)\,Q^\top\big] \geq 0$ for all $0 \leq s \leq t$. That means, the long-term yield is only dependent upon the stochastic long-term drift. This outcome extends a result stated in Section 2.2 of [38]. Here the drift is stochastic since μ_∞ is given by (4.3), hence it depends on (the limit of) the volatility and on X. As in [38], it is still true that the form (4.31) of ℓ remains the same under a change of equivalent probability measures. This can be proven by applying the convergence results of Propositions 4.4 and 4.5 to the yield expressed in the form (3.42), which yields the representation for Y under a change of equivalent probability measures.

Further, it follows immediately from Lemma 4.2 and Lemma 4.3 that $(\ell_t)_{t\geq 0}$ is a non-decreasing process what was shown for the first time in 1996 by Dybvig, Ingersoll and Ross in [22] and generally proven in [35].

To conclude we now discuss some conditions on the volatility process $\sigma(t,T)$ that guarantee the existence of the long-term drift μ_∞.

Proposition 4.6. *Let $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{\sqrt{T-t}}\Big)$ for every $t \geq 0$, i.e. $\sigma(t,T)_{ij} \in \mathrm{O}\Big(\frac{1}{\sqrt{T-t}}\Big)$ for all $i,j \in \{1,\dots,d\}$ $\mathbb{P}$-a.s. Under the setting outlined in* Section 3, *we get*

$$\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] < \infty \;\; \mathbb{P}\text{-}a.s.$$

Proof. Let $t \geq 0$ and $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{\sqrt{T-t}}\Big)$. Then

$$\begin{aligned}
\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] &\overset{(4.30)}{=} \lim_{T\to\infty}\frac{1}{T-t}\left\|\sqrt{X_t}\,\Sigma(t,T)\,Q^\top\right\|^2 \\
&\overset{(3.3)}{=} \lim_{T\to\infty}\frac{1}{T-t}\sum_{i,j,k,l,m} Q_{ij}\int_t^T \sigma(t,u)_{jk}du\; X_{kl,t}\int_t^T \sigma(t,u)_{lm}du\; Q^\top_{mi} \\
&< \;\; \infty \;\; \mathbb{P}\text{-a.s.}
\end{aligned}$$

□

Proposition 4.7. *Let $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{T-t}\Big)$ for every $t \geq 0$ $\mathbb{P}$-a.s. Under the setting outlined in* Section 3, *we get*

$$\mu_\infty(t) = 0$$

and therefore $(\ell_t)_{t\geq 0}$ is constant.

Proof. Let $t \geq 0$ and $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{T-t}\Big)$, i.e. for all $i,j \in \{1,\dots,d\}$ it is $\sigma(t,T)_{ij} \in \mathrm{O}\Big(\frac{1}{T-t}\Big)$. Then, we get for all $i,j \in \{1,\dots,d\}$ that $\Sigma(t,T)_{ij} \in \mathrm{O}(\log(T-t))$ and therefore for all $i,j,k,l \in \{1,\dots,d\}$ that

$$\lim_{T\to\infty}\frac{1}{T-t}\,\Sigma(t,T)_{ij}\;\Sigma(t,T)_{kl} = 0 \;\; \mathbb{P}\text{-a.s.} \tag{4.32}$$

Hence, for all $i,j \in \{1,\dots,d\}$ it is

$$\mu_\infty(t)_{ij} \overset{(4.3)}{=} \lim_{T\to\infty}\frac{\Gamma(t,T)_{ij}}{T-t} \overset{(3.38)}{=} \lim_{T\to\infty}\frac{1}{T-t}\sum_{k,l}\Sigma(t,T)_{ik}\,X_{kl,t}\Sigma(t,T)_{lj} \overset{(4.32)}{=} 0 \;\; \mathbb{P}\text{-a.s.}$$

By (4.31) this yields $\ell_t = \ell_0$ for all $t \geq 0$, i.e. $(\ell_t)_{t\geq 0}$ is constant. □

The following table summarises the results regarding the convergence behaviour of the long-term yield for all $t \geq 0$.

Long-term drift	Long-term volatility	Long-term yield	Volatility curve
$\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] = \infty$	$0 < \lVert\sigma_\infty(t)\rVert < \infty$	infinite	$\sigma(t,T) \sim \mathrm{O}(1)$
$\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] = \infty$	$0 < \lVert\sigma_\infty(t)\rVert < \infty$	infinite	$\sigma(t,T) \sim \mathrm{O}(T-t)$
$\mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] = 0$	$\lVert\sigma_\infty(t)\rVert = 0$	constant	$\sigma(t,T) \sim \mathrm{O}\Big(\frac{1}{T-t}\Big)$
$0 < \mathrm{Tr}\big[Q\,\mu_\infty(t)\,Q^\top\big] < \infty$	$\lVert\sigma_\infty(t)\rVert = 0$	non-decreasing	$\sigma(t,T) \sim \mathrm{O}\Big(\frac{1}{\sqrt{T-t}}\Big)$

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